

Conjectures for Large N $\mathcal{N} = 4$ Superconformal Chiral Primary Four Point Functions

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An expression for the four point function for $\frac{1}{2}$ -BPS operators belonging to the $[0, p, 0]$ $SU(4)$ representation in $\mathcal{N} = 4$ superconformal theories at strong coupling in the large N limit is suggested for any p . It is expressed in terms of the four point integrals defined by integration over AdS_5 and agrees with, and was motivated by, results for $p = 2, 3, 4$ obtained via the AdS/CFT correspondence. Using crossing symmetry and unitarity, the detailed form is dictated by the requirement that at large N the contribution of long multiplets with twist less than $2p$, which do not have anomalous dimensions, should cancel corresponding free field contributions.

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1. Introduction

The discovery of the AdS/CFT correspondence has demonstrated the intimate relationship between string theory and quantum field theory. In particular the initial proposal of an essential equivalence of type IIB string theory on $AdS_5 \times S^5$ with the maximal $\mathcal{N} = 4$ superconformal gauge theories on the four dimensional boundary has proved especially fruitful. The isometry group of $AdS_5 \times S^5$, $SO(4, 2) \times SO(6) \simeq SU(2, 2) \times SU(4)$, matches precisely the bosonic part of the $\mathcal{N} = 4$ superconformal group $PSU(2, 2|4)$. For gauge group $SU(N)$ and a gauge coupling g then at large N the supergravity approximation to ten dimensional string theory may be used to determine the strong coupling, $\lambda = g^2 N \rightarrow \infty$, behaviour of the $\mathcal{N} = 4$ superconformal gauge theory. As shown by Witten [1] results for the leading large N behaviour of correlation functions of gauge invariant operators can be calculated by considering Feynman graphs involving propagators on AdS_5 linking points on the AdS_5 boundary to supergravity vertices in the bulk.

For $\mathcal{N} = 4$ superconformal theories the simplest operators to discuss are the so called chiral primary operators which are annihilated by half the supercharges and so are part of short $\frac{1}{2}$ -BPS multiplets. These are scalars belonging to the $SU(4)_R$ representation with Dynkin labels $[0, p, 0]$ and have a protected scale dimension $\Delta = p$. They are simply related to Kaluza-Klein modes in the expansion of supergravity fields on S^5 . The three point functions of such operators were determined by Lee *et al* [2], see also [3]. In this case the x -dependence is determined by conformal invariance apart from an overall constant. The normalisation constant calculated for large N using supergravity in the strong coupling limit is identical to the result for free field theory for large N , which corresponds to the expectation that the three point functions for chiral primary operators are independent of g [4,5]. The corresponding four point functions however have a non trivial dependence on the coupling, although this is also constrained by non renormalisation theorems [6,7] and superconformal Ward identities [8,9,10]. This is reflected by the fact that the operator product expansion allows for the presence of contributions from long multiplets which have anomalous dimensions expressible perturbatively as an expansion in g .

Using the AdS/CFT correspondence the basic building blocks for n -point correlation functions are, as defined in [11], given by integrals on AdS_{d+1} of the form

$$D_{\Delta_1 \dots \Delta_n}(x_1, \dots, x_n) = \frac{1}{\pi^{\frac{1}{2}d}} \int_0^\infty dz \int d^d x \frac{1}{z^{d+1}} \prod_{i=1}^n \left(\frac{z}{z^2 + (x - x_i)^2} \right)^{\Delta_i}, \quad (1.1)$$

where x_i are points on the boundary S^d of AdS_{d+1} . The n -point functions defined by (1.1) transform covariantly under conformal transformations on the x_i with corresponding scale dimension Δ_i . Manifestly they are symmetric under permutations of x_i, Δ_i . For

$n = 3$ the integral can be evaluated to give the standard conformal form for the three point function for operators of scale dimension Δ_i . For $n = 4$ it may be reduced to a function $\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u, v)$, independent of d , of two conformal invariants u, v . In addition to relations inherited from the permutation symmetry there are various non trivial identities relating \overline{D} -functions with Δ_i differing by integers and $\sum \Delta_i$ varying by ± 2 , [11,12]. Calculations of amplitudes on AdS_5 with four points on the boundary, involving scalar, vector and graviton exchanges, may be reduced to linear combinations $D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1, x_2, x_3, x_4)$ for various integer Δ_i , and also $\sum_i \Delta_i$ even [13].

These methods then allow the determination of the large N strong coupling limit for chiral primary operator four point functions. The simplest case, and the first which was determined, is the four point function for $[0, 2, 0]$ $\frac{1}{2}$ -BPS operators [14]. These highly non trivial calculations, which require expanding supergravity to fourth order in fluctuations around $AdS_5 \times S^5$ [15,16,17], were later extended to obtain the four point functions for chiral primary operators belonging to the $[0, p, 0]$ representation for $p = 3, 4$ [18,19]. The supergravity results generate contributions from disconnected diagrams which are $O(1)$ in the large N limit and which correspond to the disconnected pieces in free field theory and also, from connected diagrams, $O(1/N^2)$ pieces which reproduce the results of free field theory to this order. There is in addition a dynamical contribution which is a linear combination of \overline{D} -functions. Using \overline{D} -identities is necessary to show that these results are in accord with the consequences of superconformal symmetry. For $p = 2$, superconformal symmetry requires that the dynamical part may be reduced to a single crossing symmetric function of u, v and, although not initially evident this can be simplified to a single \overline{D} function, $\overline{D}_{2422}(u, v)$, which is essentially crossing symmetric as a consequence of $\overline{D}_{2422}(u, v) = \overline{D}_{2422}(v, u) = \overline{D}_{2422}(u/v, 1/v)/v^4$. For $p = 3, 4$ the dynamical part reduces respectively to 1,2 functions of u, v together with their transformations related by crossing (the group of crossing transformations on 4-point functions \mathcal{S}_3 is here generated by $u \leftrightarrow v$ and $u \rightarrow u/v, v \rightarrow 1/v$). The number of \overline{D} -functions in the final answer proliferate. Nevertheless the results for $p = 3, 4$ can be reduced to expressions in which they are all just of the form $\overline{D}_{i\,p+2\,jk}$ for various $i, j, k \leq p$ [9].

The \overline{D} functions have a representation as a series expansion in powers of $u, 1 - v$ but in which terms proportional to $\ln u$ are also present. The part involving $\ln u$ is interpreted as arising from the leading term in the $1/N$ expansion of the anomalous dimensions of long multiplets whose contributions involve factors u^Δ . Anomalous dimensions are possible only for long multiplets which are here denoted by $\mathcal{A}_{nm,\ell}^\Delta$, where the lowest dimension operator belongs to a $SU(4)$ representation having Dynkin labels $[n - m, 2m, n - m]$ with also scale dimension Δ , spin ℓ . In the operator product expansion of the four point function for $[0, p, 0]$ chiral primary operators long multiplets may only be present for $m \leq n \leq p - 2$

[20,21]. This also follows from superconformal Ward identities [9]. The conformal partial wave analysis of the strong coupling results for four point functions demonstrates that anomalous dimensions are obtained [22,23,18,24] only for long multiplets $\mathcal{A}_{nm,\ell}^\Delta$ which have twist $\Delta - \ell \geq 2p$. However the unitarity bound in superconformal representation theory requires just $\Delta - \ell \geq 2n + 2$. Assuming all long multiplets necessarily have non zero anomalous dimensions to leading order in $1/N$ these results then require that long multiplets with twist $\Delta - \ell < 2p$ are absent in the operator product expansion for two $[0, p, 0]$ chiral primary operators in the large N limit.

Contributions without anomalous dimensions correspond to operators belonging to short \mathcal{B}_{nm} or semi-short superconformal multiplets $\mathcal{C}_{nm,\ell}$ whose scale dimensions are protected, the lowest dimension operator belonging to the $SU(4)$ representation $[n-m, 2m, n-m]$ has $\Delta = n$, with $\ell = 0$, or $\Delta = n + 2 + \ell$ respectively. The semi-short multiplets are related to the decomposition of long multiplets at the unitarity threshold [25]

$$\mathcal{A}_{nm,\ell}^{2n+\ell+2} \simeq \mathcal{C}_{nm,\ell} \oplus \mathcal{C}_{n+1\,m,\ell-1} \oplus \dots, \quad 0 \leq m \leq n, \quad (1.2)$$

neglecting two additional semi-short multiplets which do not contribute to the operator product expansion of two $[0, p, 0]$ chiral primary operators. The relation (1.2) extends to $\ell = 0$ if we identify [25]

$$\mathcal{C}_{nm,-1} \simeq \mathcal{B}_{n+1\,m}, \quad (1.3)$$

where for $n > m$ \mathcal{B}_{nm} is a $\frac{1}{4}$ -BPS multiplet. Only such short or semi-short multiplets therefore contribute to the operator product expansion of the four point function considered here in the large N limit for twist $\Delta - \ell < 2p$.

As a consequence of (1.2) there is a potential ambiguity in the decomposition of the operator product expansion into contributions from various supermultiplets in a theory in which long multiplets do not have anomalous dimensions. Following [26] if we denote $n[\mathcal{M}]$ the number of supermultiplets \mathcal{M} contributing to the operator product expansion for two $[0, p, 0]$ chiral primary operators then the index

$$I_{nm} = \sum_{\ell=-1}^{n-m} (-1)^{\ell+1} n[\mathcal{C}_{n-\ell\,m,\ell}], \quad (1.4)$$

is such that combinations of short or semi-short multiplets forming a long multiplet cancel. The number of $\frac{1}{2}$ -BPS multiplets $n[\mathcal{B}_{nn}]$ and also $n[\mathcal{B}_{n+1\,n}]$ are also invariants. In the case of interest here by crossing multiplets $\mathcal{M}_{nm,\ell}$ can only contribute if $n + m + \ell$ is even (in consequence $\frac{1}{4}$ -BPS multiplets $\mathcal{B}_{n+1\,n}$ cannot appear) so that I_{nm} is relevant only for $n + m$ even. The results obtained from analysis of the four point function for the operator product expansion for two $[0, p, 0]$ chiral primary operators to order $1/N$ is that $I_{nm} \neq 0$

for all $0 \leq m \leq n$ with $n \geq p - 2$ and $I_{nm} = 0$ for $n < p - 2$. The index is saturated by requiring just the multiplets $\mathcal{C}_{p-1\,m,\ell}$ and \mathcal{B}_{pm} to be present. There are as well $\frac{1}{2}$ -BPS multiplets \mathcal{B}_{nn} for $n = 1, \dots, p$.

From the large N results for the $p = 2$ case it was shown in [22] that the only twist two singlet operator necessary in the operator product expansion is when $\ell = 2$, corresponding to the energy momentum tensor. This was confirmed in [23] where the absence of all leading twist two singlet operators belonging to long supermultiplets was also demonstrated for any ℓ using a simplified form of the large N results. This is as expected since these operators are absent in the large N limit, their dimensions are proportional to $\lambda^{\frac{1}{4}}$ as $\lambda \rightarrow \infty$ [27] so that they decouple from the spectrum revealed by the large N operator product expansion. This is different from the perturbative expansion about free field theory when such operators are present for any ℓ ($\ell = 0$ is the Konishi scalar) and have a leading anomalous dimension proportional to λ , which is therefore not suppressed by $1/N$, whereas multi-trace operators have anomalous dimensions which are $\mathcal{O}(\lambda/N^2)$ [28]. The disappearance of twist two operators belonging to long multiplets in the strong coupling limit $\lambda \rightarrow \infty$ requires in the operator product calculations in [23] a non trivial cancellation between the free field $\mathcal{O}(1/N^2)$ contributions and also the leading non $\ln u$ terms from the dynamical \overline{D} -functions. These cancellations were shown to extend also to the $p = 3, 4$ cases in [24]. The operator product expansion for large N strong coupling then has contributions only from multi-trace operators with anomalous dimension suppressed by $1/N^2$.

We use this cancellation in this paper as a guiding principle to determine an expression for the four point function of single trace $[0, p, 0]$ chiral primary operators for any p . The free field results are straightforward to obtain to order $1/N^2$. Although there are many, $[\frac{1}{12}p(p+6)] + 1$, crossing symmetric forms, each of which may in general have a different multiplicative constant depending on N in a complicated fashion, as $N \rightarrow \infty$ then these simplify to just $\frac{p^2}{N^2}$ or $\frac{2p^2}{N^2}$, assuming the coefficients of disconnected contributions are normalised to 1. The factor p^2 reflects the cyclic symmetry of the trace for single trace operators. The dynamical part at large N is assumed to be expressible as a linear combination of \overline{D} -functions. In order to accommodate the consequences of $SU(4)_R$ symmetry we introduce, instead of rank p symmetric traceless tensors for each $[0, p, 0]$ chiral primary field, variables σ, τ so that the four point amplitude becomes a polynomial in σ, τ of degree p . The number of independent terms corresponds exactly to the number of invariants formed by four $[0, p, 0]$ representations, or the number of irreducible representations in the tensor product $[0, p, 0] \otimes [0, p, 0] \simeq \bigoplus_{m=0, \dots, n, n=0, \dots, p} [n-m, 2m, n-m]$. The operator product expansion for four point function is then equivalent to a simultaneous expansion in terms of harmonic functions of σ, τ for $SO(6)$ and also in harmonic functions of u, v for

the non compact group $SO(4, 2)$. Crossing symmetry transformations are now generated by $u \leftrightarrow v$, $\sigma \rightarrow \sigma/\tau$, $\tau \rightarrow 1/\tau$, and $u \rightarrow u/v$, $v \rightarrow 1/v$, $\sigma \leftrightarrow \tau$. With the aid of superconformal symmetry the dynamical part reduces to a crossing symmetric form \mathcal{H} which is a polynomial in σ, τ of degree $p - 2$ [9]. The operator product expansion for \mathcal{H} corresponds only to long multiplets. We determine all possible crossing symmetric expressions for this dynamical term formed from $\overline{D}_{i p+2 j k}(u, v)$ for appropriate i, j, k , which are constrained by removal of identities amongst such \overline{D} -functions and also by the requirement that in each channel unitarity constraints on operators appearing in the operator product expansion are satisfied. Although each crossing symmetric form has $\ln u$ terms which arise in the operator product expansion only for $\Delta - \ell \geq p$ there are also sub leading terms which contribute for $\Delta - \ell < p$. Our assumption is that this must be cancelled by a corresponding free contribution. Within our framework this determines all coefficients uniquely, independent of any supergravity calculations on the basis of the AdS/CFT correspondence. Of course this agrees with known results for $p = 2, 3, 4$.

In detail in the next section we define the four point correlation functions which are investigated here and introduce six dimensional $SO(6)$ null vectors in terms of which the variables σ, τ are constructed. We also give the two variable harmonic functions $Y_{nm}(\sigma, \tau)$ which allow a decomposition into $SU(4)$ representations. In section three we recapitulate expressions for the two variable conformal partial waves, which are harmonic functions u, v with a similar form to Y_{nm} , and show how the expansion can be simplified by first considering a decomposition into contributions for given twist which is described by a single variable function of a single variable x . Section four uses this and superconformal identities to show a decomposition which matches the contributions of differing superconformal multiplets in the operator product expansion. The results are equivalent to those in [9], as used in [24], but are in a form more convenient for later discussion here. Section five gives the results for free field theory, initially in terms of crossing symmetric polynomials but then in terms of the simplified form valid for large N . It is shown how contributions involving twist less than $2p$ are universal in that they do not depend on the particular p , apart from a common overall coefficient. Expressions for the universal functions of x are obtained which are rather complicated but which are simplified in some cases. In section six we analyse the possible crossing symmetric forms for \overline{D} -functions with appropriate additional constraints as described above. The general form for the dynamical part of the four point function is then supposed to be an arbitrary linear combination of the independent allowed expressions. We also exhibit those contributions from the \overline{D} -functions which play a crucial role in our discussion. These are expressed as a linear combination of hypergeometric functions of $1 - v$ and are required to have a universal form. The constraints arising from this are considered in section seven for low p and it is shown how this matches known results. A general analysis which gives a unique solution is presented in

section eight. The derivation of the results in sections 6,7,8 does not depend substantially on the rest of this paper and may be read independently. In section 9 we demonstrate the cancellation between the universal parts of the dynamical and free contributions to the four point function which is the fundamental principle behind the proposals in this paper. A few remarks are made in a final conclusion.

Some details are deferred to three appendices. In appendix A we amplify some of the results of section 4 giving detailed contributions of short and semi-short $\mathcal{N} = 4$ superconformal multiplets to the conformal partial wave expansion. In appendix B we calculate the large N limit for the free field contributions to the four point function which is used in section 5. In appendix C we prove some necessary hypergeometric identities.

2. Superconformal Correlation Functions

We here establish the essential notation for the discussion of four point correlation functions for chiral primary $\frac{1}{2}$ -BPS operators, spinless with scale dimension $\Delta = p$ and belonging to the $SU(4)$ R -symmetry $[0, p, 0]$ representation, which are the subject of interest in this paper. These chiral primary operators belong to the simplest short multiplets of $\mathcal{N} = 4$ superconformal symmetry and are represented by symmetric traceless $SO(6)$ tensor fields $\varphi_{r_1 \dots r_p}(x)$, with $r_i = 1, \dots, 6$. For detailed analysis for arbitrary p it is very convenient to consider instead $\varphi^{(p)}(x, t) = \varphi_{r_1 \dots r_p}(x) t_{r_1} \dots t_{r_p}$, homogeneous of degree p in t , for t_r an arbitrary six dimensional complex null vector. Such null vectors were employed in [18] and used in detail for simplifying superconformal transformations in [9]. The four point correlation functions of chiral primary operators may then be written in the form

$$\langle \varphi^{(p)}(x_1, t_1) \varphi^{(p)}(x_2, t_2) \varphi^{(p)}(x_3, t_3) \varphi^{(p)}(x_4, t_4) \rangle = \left(\frac{t_1 \cdot t_2 t_3 \cdot t_4}{x_{12}^2 x_{34}^2} \right)^p \mathcal{G}^{(p)}(u, v; \sigma, \tau), \quad (2.1)$$

with the definitions

$$x_{ij} = x_i - x_j, \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (2.2)$$

and σ, τ $SU(4)$ invariants which are homogeneous of degree zero and are defined by

$$\sigma = \frac{t_1 \cdot t_3 t_2 \cdot t_4}{t_1 \cdot t_2 t_3 \cdot t_4}, \quad \tau = \frac{t_1 \cdot t_4 t_2 \cdot t_3}{t_1 \cdot t_2 t_3 \cdot t_4}. \quad (2.3)$$

Necessarily, since the correlation function is homogeneous of degree p in each t_i , $\mathcal{G}^{(p)}(u, v; \sigma, \tau)$ is a polynomial of degree p in σ, τ (i.e it may be expanded in the $\frac{1}{2}(p+1)(p+2)$ monomials $\sigma^r \tau^s$ with $r+s \leq p$). Crossing symmetry requires

$$\mathcal{G}^{(p)}(u, v; \sigma, \tau) = \mathcal{G}^{(p)}(u/v, 1/v; \tau, \sigma) = \left(\frac{u}{v} \right)^p \tau^p \mathcal{G}^{(p)}(v, u; \sigma/\tau, 1/\tau). \quad (2.4)$$

As related in the introduction above, for $p = 2, 3, 4$ expressions for each term in the $SU(4)$ expansion of $\mathcal{G}^{(p)}(u, v; \sigma, \tau)$ have been obtained for large N through the AdS/CFT correspondence.

For the subsequent discussion it is necessary to consider new variables x, \bar{x} defined by

$$u = x\bar{x}, \quad v = (1-x)(1-\bar{x}). \quad (2.5)$$

In an analogous fashion to (2.5) we may also write

$$\sigma = \alpha\bar{\alpha}, \quad \tau = (1-\alpha)(1-\bar{\alpha}), \quad (2.6)$$

and $\mathcal{G}^{(p)}(u, v; \sigma, \tau)$ may be written instead in terms of a symmetric function of x, \bar{x} and also $\alpha, \bar{\alpha}$. We may decompose this into contributions for the different $SU(4)$ representations formed by the tensor product $[0, p, 0] \otimes [0, p, 0]$ by writing

$$\mathcal{G}^{(p)}(u, v; \sigma, \tau) = \sum_{0 \leq m \leq n \leq p} \mathcal{G}_{nm}^{(p)}(u, v) Y_{nm}(\sigma, \tau), \quad (2.7)$$

where Y_{nm} are two variable harmonic polynomials of degree n which correspond to the $SU(4)$ representation with Dynkin labels $[n-m, 2m, n-m]$. Explicitly we have

$$Y_{nm}(\sigma, \tau) = \frac{P_{n+1}(y)P_m(\bar{y}) - P_m(y)P_{n+1}(\bar{y})}{y - \bar{y}}, \quad y = 2\alpha - 1, \quad \bar{y} = 2\bar{\alpha} - 1, \quad (2.8)$$

with P_n the usual Legendre polynomials. Using the standard results

$$\frac{1}{2}yP_n(y) = \gamma_{n,1}P_{n+1}(y) + \gamma_{n,-1}P_{n-1}(y), \quad \gamma_{n,1} = \frac{n+1}{2(2n+1)}, \quad \gamma_{n,-1} = \frac{n}{2(2n+1)}, \quad (2.9)$$

we easily obtain the following recurrence relations which play a significant role in the later discussion,

$$\begin{aligned} (\sigma - \tau)Y_{nm}(\sigma, \tau) &= \sum_{r=\pm 1} (\gamma_{n+1,r}Y_{n+r,m}(\sigma, \tau) + \gamma_{m,r}Y_{n,m+r}(\sigma, \tau)), \\ \frac{1}{2}(\sigma + \tau - \frac{1}{2})Y_{nm}(\sigma, \tau) &= \sum_{r,s=\pm 1} \gamma_{n+1,r}\gamma_{m,s}Y_{n+r,m+s}(\sigma, \tau). \end{aligned} \quad (2.10)$$

Here we note that if appropriate we should take $Y_{n,n+1} = 0$ and $Y_{n-1,n+1} = -Y_{nn}$.

3. Conformal Partial Waves

Using the operator product expansion the four point function for scalar conformal primary fields may be expanded in terms of contributions from conformal primary operators

with scale dimension Δ and spin ℓ giving a conformal partial wave expansion. This is equivalent to an expansion in terms of functions $G_{\Delta}^{(\ell)}(u, v)$ of the two conformal invariants u, v which are essentially harmonic functions for $SO(4, 2)$ [29]. In a perturbative context we are then interested in determining the coefficients $a_{j,\ell}$ for an expansion of the form [12]

$$F(u, v) = \sum_{j,\ell=0,1,\dots} a_{j,\ell} u^j G_{2a+2j+\ell}^{(\ell)}(u, v), \quad (3.1)$$

where we require that $F(u, v)$ has a power series expansion in $u, 1-v$. In four dimensions the conformal partial waves are expressed simply in terms of the variables x, \bar{x} given by (2.5) by

$$u^j G_{2a+2j+\ell}^{(\ell)}(u, v) = - \frac{g_{a,j+\ell+1}(x) g_{a,j}(\bar{x}) - g_{a,j}(x) g_{a,j+\ell+1}(\bar{x})}{x - \bar{x}}, \quad (3.2)$$

where

$$g_{a,j}(x) = (-x)^j F(a+j-1, a+j-1; 2a+2j-2; x), \quad j = 0, 1, 2, \dots \quad (3.3)$$

which satisfies $g_{a,j+n}(x) = (-x)^n g_{a+n,j}(x)$. As a consequence of the form (3.2) we may impose

$$a_{j,\ell} = -a_{j+\ell+1, -\ell-2}, \quad a_{j,-1} = 0, \quad (3.4)$$

and then the expansion (3.1) may be written in the form

$$(x - \bar{x}) F(u, v) = \sum_{j=0}^{\infty} F_j(x) g_{a,j}(\bar{x}), \quad (3.5)$$

where

$$F_j(x) = \sum_{\ell=0}^{j-1} a_{j-\ell-1,\ell} g_{a,j-\ell-1}(x) - \sum_{\ell=0}^{\infty} a_{j,\ell} g_{a,j+\ell+1}(x) = - \sum_{\ell=-j-1}^{\infty} a_{j,\ell} g_{a,j+\ell+1}(x). \quad (3.6)$$

As special cases we may note that

$$\begin{aligned} G_{\ell}^{(\ell)}(u, v) &= - \frac{(1 - \frac{1}{2}\bar{x}) g_{0,\ell+1}(x) - (1 - \frac{1}{2}x) g_{0,\ell+1}(\bar{x})}{x - \bar{x}}, \\ u G_{\ell+2}^{(\ell)}(u, v) &= \frac{\bar{x} g_{0,\ell+2}(x) - x g_{0,\ell+2}(\bar{x})}{x - \bar{x}} = -u \frac{g_{1,\ell+1}(x) - g_{1,\ell+1}(\bar{x})}{x - \bar{x}}. \end{aligned} \quad (3.7)$$

For applications in the subsequent discussion the required expansions can in general be reduced to considering the case $a = 1$ when (3.3) satisfies

$$g_{1,j}(x) = (-1)^j g_{1,j}(x') \quad x' = \frac{x}{x-1}. \quad (3.8)$$

We then consider single variable expansions of the form

$$F(x) = \sum_{\ell} a_{\ell} g_{1,\ell+1}(x), \quad (3.9)$$

where $F(x)$ is analytic in x save for a branch cut along the real axis from 1 to ∞ . The coefficients a_{ℓ} may be calculated by finding a representation of the form

$$F(x) = \sum_{n=0}^N s_n x^{n+1} + \sum_{n=0}^{N'} s'_n x'^{n+1} + \int_0^1 dt r(t) \frac{x}{1-tx}, \quad (3.10)$$

where $r(t)$ is determined by the discontinuity of F across the branch cut. We may then note that

$$x^{n+1} = -(-1)^n \sum_{\ell \geq n} \frac{\ell!^2}{(2\ell)!} \frac{(\ell+n)!}{n!^2(\ell-n)!} g_{1,\ell+1}(x), \quad (3.11)$$

with a corresponding expansion for x'^{n+1} as implied by (3.8). Furthermore using¹

$$\frac{x}{1-tx} = - \sum_{\ell=0}^{\infty} \frac{\ell!^2}{(2\ell)!} (-1)^{\ell} P_{\ell}(2t-1) g_{1,\ell+1}(x), \quad (3.12)$$

where P_{ℓ} are Legendre polynomials, in the integral involving $r(t)$ in (3.9) we may determine its contribution in terms of the integrals $\int_0^1 dt r(t) P_{\ell}(2t-1)$ for $\ell = 0, 1, 2, \dots$. By expanding in t it is easy to see that (3.12) is equivalent to (3.11).

4. Superconformal Ward Identities and Operator Product Expansion

The analysis of superconformal Ward identities is independent of p so this is suppressed here. In terms of the variables defined by (2.5) and (2.6) these require [9]

$$\mathcal{G}(u, v; \sigma, \tau) \Big|_{\bar{\alpha}=\frac{1}{x}} = F(x, \alpha) = k + \left(\alpha - \frac{1}{x}\right) f(x, \alpha). \quad (4.1)$$

Although not essential from the viewpoint of superconformal symmetry dynamical requirements ensure that the function $f(x, \alpha)$ is identical to the result obtained for free fields. As a consequence of (2.4) we have, with x' as in (3.8),

$$f(x, \alpha) = -f(x', 1-\alpha). \quad (4.2)$$

¹ Since $\frac{\ell!^2}{(2\ell)!} g_{1,\ell+1}(x) = (-1)^{\ell+1} (\ell + \frac{1}{2}) Q_{\ell}(z)$, where $z = \frac{2}{x} - 1$ and Q_{ℓ} is an associated Legendre function, this is equivalent to $\frac{1}{z-y} = \sum_{\ell} (2\ell+1) P_{\ell}(y) Q_{\ell}(z)$.

The Ward identity (4.1) may be solved by writing

$$\begin{aligned} \mathcal{G}(u, v; \sigma, \tau) &= k + \mathcal{G}_f(u, v; \sigma, \tau) \\ &\quad + (\alpha x - 1)(\alpha \bar{x} - 1)(\bar{\alpha} x - 1)(\bar{\alpha} \bar{x} - 1) \mathcal{H}(u, v; \sigma, \tau), \end{aligned} \quad (4.3)$$

where the contribution from $f(x, \alpha)$ is given by

$$\begin{aligned} &\mathcal{G}_f(u, v; \sigma, \tau) + 2k \\ &= \frac{(\bar{\alpha} x - 1)(\alpha \bar{x} - 1)(F(x, \alpha) + F(\bar{x}, \bar{\alpha})) - (\alpha x - 1)(\bar{\alpha} \bar{x} - 1)(F(x, \bar{\alpha}) + F(\bar{x}, \alpha))}{(x - \bar{x})(\alpha - \bar{\alpha})}, \end{aligned} \quad (4.4)$$

with F expressed in terms of f as in (4.1), the contribution involving k cancels. Corresponding to $\mathcal{G}^{(p)}$ then $\mathcal{H}^{(p)}(u, v; \sigma, \tau)$, as defined by (4.3), is a polynomial in σ, τ of degree $p - 2$ and $f^{(p)}(x, \alpha)$ is a polynomial in α of degree $p - 1$. We may also note that

$$\begin{aligned} &(\alpha x - 1)(\alpha \bar{x} - 1)(\bar{\alpha} x - 1)(\bar{\alpha} \bar{x} - 1) \\ &= v + \sigma^2 uv + \tau^2 u + \sigma v(v - 1 - u) + \tau(1 - u - v) + \sigma \tau u(u - 1 - v). \end{aligned} \quad (4.5)$$

Non trivial dynamical contributions are only contained in $\mathcal{H}(u, v; \sigma, \tau)$.

The representation (4.3) matches exactly with the results of the operator product expansion and the requirements of superconformal symmetry. The conformal partial wave expansion for $\mathcal{G}(u, v; \sigma, \tau)$ has in general the form

$$\mathcal{G}(u, v; \sigma, \tau) = \sum_{nm, t_\ell \ell} a_{nm, t_\ell \ell} u^{t_\ell} G_{2t_\ell + \ell}^{(\ell)}(u, v) Y_{nm}(\sigma, \tau), \quad (4.6)$$

so that $a_{nm, t_\ell \ell}$ correspond to the contributions of operators belonging to the $SU(4)$ representation with Dynkin labels $[n - m, 2m, n - m]$ and with spin ℓ and scale dimension $\Delta = 2t_\ell + \ell$. For theories with superconformal symmetry the operators appearing in the operator product expansion must belong to supermultiplets each of which must have the same anomalous dimension. For $\mathcal{N} = 4$ superconformal the supermultiplets are long $\mathcal{A}_{nm, \ell}^\Delta$, which may have an anomalous dimension depending on the coupling, and semi-short $\mathcal{C}_{nm, \ell}$ or short \mathcal{B}_{nm} , where the scale dimensions are protected against perturbative corrections. For each possible $\mathcal{N} = 4$ supermultiplet which contributes to the operator product expansion the superconformal primary operator with the lowest scale dimension Δ satisfies

$$\Delta_{\text{BPS}} = 2n, \quad \Delta_{\text{semi}} = 2n + 2 + \ell, \quad \Delta_{\text{long}} \geq 2n + 2 + \ell. \quad (4.7)$$

There are also non unitary semi-short supermultiplets with $\Delta_{\text{semi}} = 2m + \ell$. These play a role in the superconformal decomposition of the operator product expansion [9], but

their contributions are cancelled in unitary theories. If a similar expansion to (4.6) for $\mathcal{H}(u, v; \sigma, \tau)$,

$$\mathcal{H}(u, v; \sigma, \tau) = \sum_{nm, j_\ell \ell} A_{nm, j_\ell \ell} u^{j_\ell} G_{2j_\ell + 4 + \ell}^{(\ell)}(u, v) Y_{nm}(\sigma, \tau), \quad (4.8)$$

is substituted in (4.3), then for each term involving $A_{nm, j_\ell \ell}$ the factor (4.5) generates a set of contributions for $a_{nm, t_\ell \ell}$ which correspond to the long supermultiplet $\mathcal{A}_{nm, \ell}^{\Delta_\ell}$ with the lowest state belonging to the representation $[n - m, 2m, n - m]$ and spin ℓ and scale dimension $\Delta_\ell = 2j_\ell + \ell$, all with the same anomalous dimension. Detailed expressions are contained in [9]. Thus \mathcal{H} determines the spectrum of all long multiplets which may contribute as conformal partial waves in the operator product expansion for the four point function (2.1). For $\mathcal{H}^{(p)}$ in (4.8) necessarily we must require $n \leq p - 2$.

For free theories, which are the starting point for perturbative treatments, there are no anomalous dimensions and in (4.6), (4.8) we may restrict $t_\ell = t$, $j_\ell = j$ to be integers. It is convenient to first consider an expansion in terms of single variable functions, following (3.5), so that consequence we write

$$(x - \bar{x})\mathcal{G}(u, v; \sigma, \tau) = x \sum_{t=0}^{\infty} \mathcal{G}_t(x, \sigma, \tau) g_{0,t}(\bar{x}), \quad (4.9)$$

and then the complete partial wave expansion is obtained from

$$\mathcal{G}_t(x, \sigma, \tau) = \sum_{\ell=-t-1}^{\infty} \sum_{nm} a_{nm, t\ell} g_{1, t+\ell}(x) Y_{nm}(\sigma, \tau). \quad (4.10)$$

In a similar fashion we write

$$(x - \bar{x})\mathcal{H}(u, v; \sigma, \tau) = \frac{1}{x} \sum_{j=0}^{\infty} \mathcal{H}_j(x, \sigma, \tau) g_{2,j}(\bar{x}), \quad (4.11)$$

with correspondingly,

$$\mathcal{H}_j(x, \sigma, \tau) = \sum_{\ell=-j-1}^{\infty} \sum_{nm} A_{nm, j\ell} g_{1, j+\ell+2}(x) Y_{nm}(\sigma, \tau). \quad (4.12)$$

For consistency (2.4) requires, with x' as in (4.2), since $g_{1,\ell}(x) = (-1)^\ell g_{1,\ell}(x')$,

$$\mathcal{G}_t(x, \sigma, \tau) = (-1)^t \mathcal{G}_t(x', \tau, \sigma), \quad \mathcal{H}_j(x, \sigma, \tau) = (-1)^j \mathcal{H}_j(x', \tau, \sigma). \quad (4.13)$$

In general for any supermultiplet \mathcal{M} we have a conformal partial wave expansion

$$\mathcal{G}_t(\mathcal{M}; x, \sigma, \tau) = \sum_{nm, \ell} a_{nm, t\ell}(\mathcal{M}) g_{1, t+\ell}(x) Y_{nm}(\sigma, \tau), \quad (4.14)$$

where the non zero $a_{nm,t\ell}(\mathcal{M})$ correspond to the spectrum of operators in the supermultiplet \mathcal{M} . Any result for $a_{nm,t\ell}(\mathcal{M})$ must be compatible with the relation from (3.4) $a_{nm,t\ell} = -a_{nm,t+\ell+1-\ell-2}$.

Given the decomposition shown in (4.3) and (4.4) we may write

$$\mathcal{G}_t(x, \sigma, \tau) = k \mathcal{G}_t(\mathcal{I}; x) + \mathcal{G}_{f,t}(x, \sigma, \tau) + \mathcal{G}_{\mathcal{H},t}(x, \sigma, \tau). \quad (4.15)$$

To obtain $\mathcal{G}_{\mathcal{H},t}$ we use

$$(\alpha\bar{x} - 1)g_{a+1,j}(\bar{x}) = -g_{a,j}(\bar{x}) - (\alpha - \frac{1}{2})g_{a,j+1}(\bar{x}) - c_{j+a}g_{a,j+2}(\bar{x}), \quad (4.16)$$

where

$$c_j = \gamma_{j,-1}\gamma_{j-1,1} = \frac{j^2}{4(2j-1)(2j+1)}, \quad (4.17)$$

to obtain $(\alpha\bar{x} - 1)(\bar{\alpha}\bar{x} - 1)g_{2,j}(\bar{x}) = \sum_t c_{t,j} g_{0,t}(\bar{x})$ with

$$\begin{aligned} c_{j,j} &= 1, \quad c_{j+1,j} = \frac{1}{2}(y + \bar{y}), \quad c_{j+2,j} = \frac{1}{4}y\bar{y} + c_j + c_{j+1}, \\ c_{j+3,j} &= c_{j+1}\frac{1}{2}(y + \bar{y}), \quad c_{j+4,j} = c_{j+1}c_{j+2}, \end{aligned} \quad (4.18)$$

where $\frac{1}{2}(y + \bar{y}) = \sigma - \tau$ and $\frac{1}{4}y\bar{y} = \frac{1}{2}(\sigma + \tau - \frac{1}{2})$. With these results from (4.9) and (4.11)

$$\mathcal{G}_{\mathcal{H},t}(x, \sigma, \tau) = \left(\alpha - \frac{1}{x}\right)\left(\bar{\alpha} - \frac{1}{x}\right)\sum_{j \geq 0} c_{t,j} \mathcal{H}_j(x, \sigma, \tau). \quad (4.19)$$

The expansions (4.10) and (4.12) may then be related with the aid of

$$\left(\alpha - \frac{1}{x}\right)\left(\bar{\alpha} - \frac{1}{x}\right)g_{1,j+1}(x) = \sum_t c_{t,j} g_{1,t-1}(x). \quad (4.20)$$

The contribution of a long multiplet $\mathcal{A}_{nm,k}^{2j+k}$ is defined by letting in (4.12) $\mathcal{H}_j(x, \sigma, \tau) \rightarrow g_{1,j+\ell+2}(x) Y_{nm}(\sigma, \tau)$ so that (4.19) and (4.14) give, using (4.20),

$$\sum_{rs} a_{rs,t\ell}(\mathcal{A}_{nm,k}^{2j+k}) Y_{rs} = c_{t+\ell+1,j+k+1} c_{t,j} Y_{nm}, \quad (4.21)$$

where we must have $j \leq t \leq j+4$ and $j+k \leq t+\ell \leq j+k+4$. Detailed expressions are easily obtained using (4.18) and (2.10). Thus $\mathcal{H}_j(x, \sigma, \tau)$ determines the contribution of all long multiplets whose lowest operator has twist $2j$.

In (4.15) $\mathcal{G}_t(\mathcal{I})$ corresponds to the contribution for the unit operator and is determined by

$$x - \bar{x} = x \sum_{t=0}^1 \mathcal{G}_t(\mathcal{I}; x) g_{0,t}(\bar{x}), \quad (4.22)$$

which gives

$$\mathcal{G}_0(\mathcal{I}; x) = 1, \quad \mathcal{G}_1(\mathcal{I}; x) = \frac{1}{2} \left(\frac{1}{x} - \frac{1}{x'} \right) = -g_{1,-1}(x). \quad (4.23)$$

In this case we then have only the non zero contributions

$$a_{00,00}(\mathcal{I}) = -a_{00,1-2}(\mathcal{I}) = 1. \quad (4.24)$$

To analyse the contributions arising from $f(x, \alpha)$, as expressed in $\mathcal{G}_{f,t}$, we use the expansion

$$f(x, \alpha) = \sum_{\ell=0}^{\infty} f_{\ell}(\alpha) g_{1,\ell+1}(x) = \sum_{n,\ell} b_{n,\ell} P_n(y) g_{1,\ell+1}(x), \quad (4.25)$$

where $f_{\ell}(\alpha) = (-1)^{\ell} f_{\ell}(1 - \alpha)$ and as shown later $f_{\ell}(\alpha) = \mathcal{O}(\alpha^{\ell})$, unless otherwise constrained by the value of p . This ensures that $b_{n,\ell}$ is non zero only if $n \leq \ell, p - 1$ and $n + \ell$ even. Using (4.9) for \mathcal{G}_f , as given by (4.4), with (4.25) then gives

$$\begin{aligned} \mathcal{G}_{f,t}(x, \sigma, \tau) = & - \left(\alpha - \frac{1}{x} \right) \left(\bar{\alpha} - \frac{1}{x} \right) \left(\frac{f(x, \alpha) - f(x, \bar{\alpha})}{\alpha - \bar{\alpha}} \delta_{t0} + \frac{y f(x, \alpha) - \bar{y} f(x, \bar{\alpha})}{2(\alpha - \bar{\alpha})} \delta_{t1} \right) \\ & + \sum_{\ell \geq 0} c_{t,\ell} \frac{\left(\alpha - \frac{1}{x} \right) f_{\ell}(\alpha) - \left(\bar{\alpha} - \frac{1}{x} \right) f_{\ell}(\bar{\alpha})}{\alpha - \bar{\alpha}}. \end{aligned} \quad (4.26)$$

If we define

$$f(x, \alpha) = f_0(x, \alpha) + b_{0,0} g_{1,1}(x), \quad (4.27)$$

and then with this decomposition (4.26) can be expressed in the form

$$\mathcal{G}_{f,t}(x, \sigma, \tau) = b_{0,0} \mathcal{G}_t(\mathcal{I}; x) - \frac{1}{6} b_{0,0} \mathcal{G}_t(\mathcal{B}_{11}; x, \sigma, \tau) + \mathcal{G}_{f_0,t}(x, \sigma, \tau), \quad (4.28)$$

where

$$\frac{1}{6} \mathcal{G}_t(\mathcal{B}_{11}; x, \sigma, \tau) = \begin{cases} \sum_{\ell=2}^4 c_{\ell,0} g_{1,\ell-1}(x), & t = 1, \\ -c_{t,0}, & t = 2, 3, 4. \end{cases} \quad (4.29)$$

Using $c_{2,0} = \frac{1}{6} Y_{11}$, $c_{3,0} = \frac{1}{36} Y_{10}$ it is straightforward to see that in the conformal partial wave expansion of $\mathcal{G}_{\mathcal{B}_{11}}$, as in (4.14), this gives just the non zero results for $a_{nm,t\ell}(\mathcal{B}_{11})$,

$$a_{11,10}(\mathcal{B}_{11}) = 1, \quad a_{10,11}(\mathcal{B}_{11}) = \frac{1}{6}, \quad a_{00,12}(\mathcal{B}_{11}) = \frac{1}{30}, \quad (4.30)$$

apart from those related by the symmetry (3.4). This corresponds to the contribution of the lowest $\frac{1}{2}$ -BPS short multiplet \mathcal{B}_{11} containing the energy momentum tensor.

In a free theory, where there are no anomalous dimensions, there is no direct distinction between the contribution of operators belonging to long multiplet or

semi-short multiplets which have protected scale dimensions. This reflects the the potential decomposition of the contribution of a long multiplet at the unitarity threshold into two corresponding semi-short multiplets as in (1.2). Such ambiguities in the analysis of the operator product expansion must be resolved in a perturbative analysis of anomalous dimensions.

To discuss the form of the contribution of semi-short multiplets we first identify for a function $\mathcal{F}(x, \sigma, \tau)$ an expression $\mathcal{H}_{\mathcal{F}}^{(j)}$, representing contributions with just twist j , which is determined by

$$\begin{aligned}\mathcal{H}_{\mathcal{F},k}^{(j)}(x, \sigma, \tau) &= \delta_{kj} \mathcal{F}(x, \sigma, \tau) - \mathcal{F}_k(\sigma, \tau) g_{1,j+1}(x), \\ \mathcal{F}(x, \sigma, \tau) &= \sum_n \mathcal{F}_n(\sigma, \tau) g_{1,n+1}(x).\end{aligned}\tag{4.31}$$

Here we must have $\mathcal{F}(x, \sigma, \tau) = (-1)^j \mathcal{F}(x', \tau, \sigma)$. Correspondingly we define

$$\mathcal{G}_{\mathcal{F},t}^{(j)}(x, \sigma, \tau) = \delta_{tj} \left(\alpha - \frac{1}{x} \right) \left(\bar{\alpha} - \frac{1}{x} \right) \mathcal{F}(x, \sigma, \tau) - \sum_{k=0}^t c_{t,k} \mathcal{F}_k(\sigma, \tau) g_{1,j-1}(x).\tag{4.32}$$

The right hand sides of (4.31) and (4.32) are such that in the expansions (4.12) and (4.10) they are compatible with the conditions $A_{nm,j\ell} = -A_{nm,j+\ell+1-\ell-2}$ and $a_{nm,t\ell} = -a_{nm,t+\ell+1-\ell-2}$, as required by (3.4).

We then define for the function $f(x, \alpha)$

$$\begin{aligned}\mathcal{Q}_j^0(x, \sigma, \tau) &= - \frac{P_j(\bar{y}) f(x, \alpha) - P_j(y) f(x, \bar{\alpha})}{\alpha - \bar{\alpha}}, \\ \mathcal{Q}_j^1(x, \sigma, \tau) &= - \frac{P_j(\bar{y}) y f(x, \alpha) - P_j(y) \bar{y} f(x, \bar{\alpha})}{2(\alpha - \bar{\alpha})},\end{aligned}\tag{4.33}$$

More generally the contribution of semi-short multiplets are determined by $f(x, \alpha)$ and expressed in terms of $\hat{\mathcal{G}}_{f,t}^{(j)}$ which is of the form, using the notation in (2.9) and (4.32),

$$\begin{aligned}\hat{\mathcal{G}}_{f,t}^{(j)}(x, \sigma, \tau) &= \mathcal{G}_{\mathcal{Q}_j^0,t}^{(j)}(x, \sigma, \tau) + \mathcal{G}_{\mathcal{Q}_j^1,t}^{(j+1)}(x, \sigma, \tau) + \gamma_{j,-1} \mathcal{G}_{\mathcal{Q}_{j-1}^0,t}^{(j+1)}(x, \sigma, \tau) \\ &\quad + \gamma_{j,-1} \mathcal{G}_{\mathcal{Q}_{j-1}^1,t}^{(j+2)}(x, \sigma, \tau) + c_j \mathcal{G}_{\mathcal{Q}_j^0,t}^{(j+2)}(x, \sigma, \tau) \\ &\quad + \gamma_{j,-1} c_{j+1} \mathcal{G}_{\mathcal{Q}_{j-1}^0,t}^{(j+3)}(x, \sigma, \tau).\end{aligned}\tag{4.34}$$

Using (4.26) we may easily recognise that

$$\mathcal{G}_{f,t}(x, \sigma, \tau) = \mathcal{G}_{\mathcal{Q}_0^0,t}^{(0)}(x, \sigma, \tau) + \mathcal{G}_{\mathcal{Q}_0^1,t}^{(1)}(x, \sigma, \tau) = \hat{\mathcal{G}}_{f,t}^{(0)}(x, \sigma, \tau).\tag{4.35}$$

If we denote by $\mathcal{C}_{nm,\ell}$ a semi-short multiplet in which the lowest state belongs to the $SU(4)$ representation $[n-m, 2m, n-m]$ and has spin ℓ , so that $\Delta - \ell = 2n + 2$, then the contributions to the partial wave expansion are determined by

$$\hat{\mathcal{G}}_{f,t}^{(j)}(x, \sigma, \tau) \Big|_{f(x, \alpha) = \frac{1}{2}P_i(y)g_{1,j+k+2}(x)} = \mathcal{G}_t(\mathcal{C}_{j-1\ i, k}; x, \sigma, \tau), \quad (4.36)$$

for $j \geq i + 1$, $k \geq 0$. In this case (4.33) gives $\mathcal{Q}_j^0 = Y_{j-1\ i} g_{1,j+k+2}$ so that it is easy to see that the contribution of the lowest dimension operator in (4.14) is given by $a_{j-1\ i, jk}(\mathcal{C}_{j-1\ i, k}) = 1$. The remaining contributions to $a_{nm, t\ell}(\mathcal{C}_{j-1\ i, k})$, with $t = j, \dots, j+3$ and $\ell = k-3, \dots, k+4$, may be easily calculated from (4.34) and (4.32) using (2.10). For application to the conformal partial wave expansion we may note that (4.36) can be extended by using,

$$\mathcal{G}_t(\mathcal{C}_{j-1\ i, -1}) = \begin{cases} \gamma_{j,1} \mathcal{G}_t(\mathcal{B}_{ji}), & j > i, \\ \gamma_{j,1} (\mathcal{G}_t(\mathcal{B}_{jj}) - \gamma_{j+1,1} \gamma_{j,1} \mathcal{G}_t(\mathcal{B}_{j+1\ j+1})), & j = i, \end{cases} \quad (4.37)$$

where the first case follows from (1.3). As in the introduction \mathcal{B}_{ji} , $j \geq i$ denotes a short multiplet whose lowest state belongs to the representation $[j-i, 2i, j-i]$ with $\Delta = 2j$. For $j > i$ this is a $\frac{1}{4}$ -BPS multiplet whereas \mathcal{B}_{jj} is a $\frac{1}{2}$ -BPS multiplet. The decomposition of $\mathcal{G}_t(\mathcal{B}_{ji})$ for $i = 0, \dots, j$, which determines the contributions for the short BPS multiplets, is exhibited in appendix A.

Furthermore using the relations (2.9) we may combine two contributions of the form (4.34) to form those for a long multiplet

$$\begin{aligned} \hat{\mathcal{G}}_{f,t}^{(j)}(x, \sigma, \tau) + \gamma_{j,1} \hat{\mathcal{G}}_{f,t}^{(j+1)}(x, \sigma, \tau) &= \left(\alpha - \frac{1}{x}\right) \left(\bar{\alpha} - \frac{1}{x}\right) \sum_k c_{t,k} \mathcal{H}_{\mathcal{Q}_j^0, k}^{(j)}(x, \sigma, \tau) \\ &= \mathcal{G}_{\mathcal{H}, t}(x, \sigma, \tau) \quad \text{for } \mathcal{H}_k = \mathcal{H}_{\mathcal{Q}_j^0, k}^{(j)}, \end{aligned} \quad (4.38)$$

using (4.19) and in accord with (4.31),

$$\begin{aligned} \mathcal{H}_{\mathcal{Q}_j^0, k}^{(j)}(x, \sigma, \tau) &= \delta_{kj} \mathcal{Q}_j^0(x, \sigma, \tau) - \mathcal{Q}_{j,k}^0(\sigma, \tau) g_{1,j+1}(x), \\ \mathcal{Q}_{j,k}^0(\sigma, \tau) &= - \frac{P_j(\bar{y}) f_k(\alpha) - P_j(y) f_k(\bar{\alpha})}{\alpha - \bar{\alpha}}. \end{aligned} \quad (4.39)$$

The relation (4.38) is a reflection of the decomposition (1.2). Using (4.38) the complete result

$$\mathcal{G}_t(x, \sigma, \tau) = k \mathcal{G}_t(\mathcal{I}; x) + \hat{\mathcal{G}}_{f,t}^{(0)}(x, \sigma, \tau) + \mathcal{G}_{\mathcal{H}, t}(x, \sigma, \tau). \quad (4.40)$$

may be rewritten by progressively removing contributions involving $\hat{\mathcal{G}}_{f,t}^{(j)}(x, \sigma, \tau)$ for $j = 0, 1, \dots$ at the expense of modifying \mathcal{H} . This reflects the inherent ambiguity in a free theory

in the decomposition into contributions from long and semi-short operators. By letting $\mathcal{H} \rightarrow \hat{\mathcal{H}}$ which is given by

$$\hat{\mathcal{H}}_k(x, \sigma, \tau) = \mathcal{H}_k(x, \sigma, \tau) + \sum_{j=0}^{J-1} (-1)^j \frac{j!^2}{(2j)!} (\delta_{kj} \mathcal{Q}_j^0(x, \sigma, \tau) - \mathcal{Q}_{j,k}^0(\sigma, \tau) g_{1,j+1}(x)), \quad (4.41)$$

then instead of (4.40) we have

$$\mathcal{G}_t(x, \sigma, \tau) = k \mathcal{G}_t(\mathcal{I}; x) + (-1)^J \frac{J!^2}{(2J)!} \hat{\mathcal{G}}_{f,t}^{(J)}(x, \sigma, \tau) + \mathcal{G}_{\hat{\mathcal{H}},t}(x, \sigma, \tau). \quad (4.42)$$

This removes the contribution of any semi-short multiplets with twist $\Delta - \ell < 2J$.

However to determine the full decomposition into supermultiplets it is necessary to separate all contributions from short multiplets. To achieve this we define, extending (4.27),

$$f(x, \alpha) = f_j(x, \alpha) + \sum_{\ell=0}^j f_\ell(\alpha) g_{1,\ell+1}(x), \quad f_\ell(\alpha) = \sum_{n=0}^{\ell} b_{n,\ell} P_n(y), \quad j < p. \quad (4.43)$$

From (4.37) we may then obtain

$$\hat{\mathcal{G}}_{f_{j-1},t}^{(j)} = \hat{\mathcal{G}}_{f_j,t}^{(j)} + 2\gamma_{j,1} \left(\sum_{i=0}^{j-1} b_{i,j} \mathcal{G}_t(\mathcal{B}_{ji}) + b_{j,j} (\mathcal{G}_t(\mathcal{B}_{jj}) - \gamma_{j,1} \gamma_{j+1,1} \mathcal{G}_t(\mathcal{B}_{j+1,j+1})) \right). \quad (4.44)$$

Using (4.44) there is a then corresponding modification of (4.38) which allows short multiplet contributions to be explicitly identified,

$$\begin{aligned} & \hat{\mathcal{G}}_{f_j,t}^{(j)}(x, \sigma, \tau) + \gamma_{j,1} \hat{\mathcal{G}}_{f_{j+1},t}^{(j+1)}(x, \sigma, \tau) \\ &= -2\gamma_{j,1} \gamma_{j+1,1} \left(\sum_{i=0}^j b_{i,j+1} \mathcal{G}_t(\mathcal{B}_{j+1,i}) + b_{j+1,j+1} (\mathcal{G}_t(\mathcal{B}_{j+1,j+1}) - \gamma_{j+1,1} \gamma_{j+2,1} \mathcal{G}_t(\mathcal{B}_{j+2,j+2})) \right) \\ &+ \left(\alpha - \frac{1}{x} \right) \left(\bar{\alpha} - \frac{1}{x} \right) \sum_k c_{t,k} (\delta_{kj} \hat{\mathcal{Q}}_j^0(x, \sigma, \tau) - \hat{\mathcal{Q}}_{j,k}^0(\sigma, \tau) g_{1,j+1}(x)), \end{aligned} \quad (4.45)$$

where since $f \rightarrow f_j$ we have $\hat{\mathcal{Q}}_j^0 = \mathcal{Q}_j^0 - \sum_{k=0}^j \mathcal{Q}_{j,k}^0$, $\hat{\mathcal{Q}}_{j,k}^0 = 0$ for $k \leq j$. Applying this result we now have

$$\begin{aligned} \mathcal{G}_t &= (k + b_{0,0}) \mathcal{G}_t(\mathcal{I}) + 2 \sum_{j=1}^{J-1} (-1)^j \frac{(j+1)!^2}{(2j+2)!} (b_{j,j} + \gamma_{j-1,1} b_{j-1,j-1}) \mathcal{G}_t(\mathcal{B}_{jj}) \\ &+ (-1)^J \frac{(J+1)!^2}{(2J+2)!} \frac{J}{2J-1} b_{J-1,J-1} \mathcal{G}_t(\mathcal{B}_{JJ}) \\ &+ 2 \sum_{j=1}^J (-1)^j \frac{(j+1)!^2}{(2j+2)!} \sum_{i=0}^{j-1} b_{i,j} \mathcal{G}_t(\mathcal{B}_{ji}) + (-1)^J \frac{J!^2}{(2J)!} \hat{\mathcal{G}}_{f_J,t}^{(J)} + \mathcal{G}_{\hat{\mathcal{H}},t}. \end{aligned} \quad (4.46)$$

Here the definition of $\hat{\mathcal{H}}$ is modified from (4.41) by $\mathcal{Q}_j^0 \rightarrow \hat{\mathcal{Q}}_j^0$ and using $f_J(x, \alpha) = \sum_{m, \ell \geq 0} b_{m, J+\ell+1} P_m(y) g_{1, J+\ell+2}(x)$ with (4.36) we have

$$\hat{\mathcal{G}}_{f_J, t}^{(J)} = 2 \sum_{m, \ell \geq 0} b_{m, J+\ell+1} \mathcal{G}_t(\mathcal{C}_{J-1} m, \ell). \quad (4.47)$$

For application to the conformal partial decomposition of chiral four point functions in the large N limit we identify $J = p$. Since in (4.47) $m = 0, \dots, p-1$ this expansion with $J = p$ gives the contribution of semi-short multiplets in the operator product expansion, unitarity requires $(-1)^p b_{m, \ell} \geq 0$ for $\ell > p$. All contributions appearing in (4.46) are then identifiable with various possible supermultiplets.

5. Free Field Results

In order to discuss the dynamical contributions it is first necessary to obtain the results for the free part of the general chiral four point function. The free field contribution satisfies the Ward identities (4.1) by requiring it to have the form

$$\mathcal{G}^{(p)}(u, v; \sigma, \tau) = \sum_{\substack{r, s \geq 0 \\ r+s \leq p}} a_{rs} (\sigma u)^r \left(\tau \frac{u}{v} \right)^s. \quad (5.1)$$

From (4.1) we then have

$$F^{(p)}(x, \alpha) = \sum_{r, s} a_{rs} (\alpha x)^r ((1 - \alpha)x')^s, \quad k = \sum_{r, s} a_{rs}. \quad (5.2)$$

However for the four point function (2.1) crossing symmetry (2.4) provides further crucial constraints on the coefficients a_{rs} . This requires that (5.1) may be rewritten in terms of two variable crossing symmetric polynomials.

Crossing symmetric polynomials in the variables σ, τ of degree n , i.e they may be expanded in monomials $\sigma^g \tau^h$ with $g + h \leq n$, are defined by

$$S^{(n)}(\sigma, \tau) = S^{(n)}(\tau, \sigma) = \tau^n S^{(n)}(\sigma/\tau, 1/\tau). \quad (5.3)$$

A simple basis for these polynomials is given by

$$S_{ab}^{(n)}(\sigma, \tau) = \begin{cases} \sigma^a \tau^a + \sigma^a \tau^{n-2a} + \sigma^{n-2a} \tau^a, & a = b, \\ \sigma^a \tau^b + \sigma^b \tau^a + \sigma^a \tau^a, & 2a + b = n, \\ \sigma^{\frac{1}{3}n} \tau^{\frac{1}{3}n}, & \frac{1}{3}n \in \mathbb{N}, \\ \sigma^a \tau^b + \sigma^b \tau^a + \sigma^a \tau^{n-a-b} + \sigma^{n-a-b} \tau^a + \sigma^b \tau^{n-a-b} + \sigma^{n-a-b} \tau^b, & \text{otherwise,} \end{cases} \quad (5.4)$$

where a, b are integers satisfying

$$0 \leq b \leq a, \quad 2a + b \leq n. \quad (5.5)$$

The first two cases in (5.4) are distinguished according to whether $n > 3a$ or $3a > n$ respectively. For any n the set of possible (a, b) are the points of an integer lattice inside or on a triangle with vertices $(0, 0)$, $(\frac{1}{3}n, \frac{1}{3}n)$ and $(\frac{1}{2}n, 0)$. The number of independent such crossing symmetric polynomials was derived in [8,9], their results are equivalent to $[\frac{1}{12}n(n+6)] + 1$. From (5.4) it is trivial to verify that the number of terms $n_{ab}^{(n)}$ in the above symmetric polynomials is in each case

$$n_{aa}^{(n)} = n_{a, n-2a}^{(n)} = 3, \quad n_{\frac{1}{3}n, \frac{1}{3}n}^{(n)} = 1, \quad n_{ab}^{(n)} = 6 \text{ otherwise}, \quad (5.6)$$

and then

$$\sum_{\substack{0 \leq b \leq a \\ \leq n-a-b}} n_{ab}^{(n)} = \frac{1}{2}(n+1)(n+2), \quad (5.7)$$

which is the number of independent polynomials in σ, τ of degree n .

We may now satisfy (2.4) by writing instead of (5.1)

$$\mathcal{G}^{(p)}(u, v; \sigma, \tau) = S_{00}^{(p)}(\sigma u, \tau u/v) + \frac{p^2}{N^2} \sum_{\substack{0 \leq b \leq a \\ \leq p-a-b}} w_{ab}^{(p)} S_{ab}^{(p)}(\sigma u, \tau u/v), \quad (5.8)$$

where $w_{00}^{(p)} = 0$ and we have imposed the normalisation condition $\mathcal{G}^{(p)}(0, v; \sigma, \tau) = 1$. In general the $w_{ab}^{(p)}$ are complicated functions of N but for single trace operators in the large N limit, as shown in appendix B, we have

$$w_{a0}^{(p)} = 1, \quad w_{ab}^{(p)} = 2, \quad b \geq 1. \quad (5.9)$$

This ensures that (5.8) may be written for any p in the form

$$\begin{aligned} \mathcal{G}^{(p)}(u, v; \sigma, \tau) = 1 + \frac{p^2}{N^2} \sum_{r=1}^{p-1} u^r \left(\sigma^r + \tau^r \frac{1}{v^r} + 2 \sum_{s=1}^{r-1} \sigma^{r-s} \tau^s \frac{1}{v^s} \right) \\ + u^p \left(\sigma^p + \tau^p \frac{1}{v^p} + \frac{p^2}{N^2} \sum_{r=1}^{p-1} \sigma^{p-r} \tau^r \frac{1}{v^r} \right), \end{aligned} \quad (5.10)$$

where the terms involving $\sigma^{r-s} \tau^s u^r$ with $0 < r < p$ are universal in that they are independent of p apart from the overall coefficient p^2/N^2 .

From (5.10) and (4.1) we easily obtain

$$k = 3 + \frac{p^2}{N^2}(p^2 - 1), \quad (5.11)$$

and

$$\mathcal{G}^{(p)}(u, v; \sigma, \tau) = 1 + \frac{p^2}{N^2} \tilde{\mathcal{G}}(u, v; \sigma, \tau) + \mathcal{O}(u^p), \quad (5.12)$$

where

$$\tilde{\mathcal{G}}(u, v; \sigma, \tau) = \frac{u\sigma + u\tau/v}{(1 - u\sigma)(1 - u\tau/v)}, \quad (5.13)$$

is a universal function, independent of p . We also then have

$$f^{(p)}(x, \alpha) = \left(k - 1 + \frac{p^2}{N^2} \left(\alpha(1 - \alpha) \frac{d^2}{d\alpha^2} + (1 - 2\alpha) \frac{d}{d\alpha} \right) \right) \frac{x}{1 - \alpha x} + \mathcal{O}(x^{p+1}). \quad (5.14)$$

Using (3.12) we easily obtain from (4.25)

$$f_\ell(\alpha) = (-1)^\ell \frac{\ell!^2}{(2\ell)!} \left(1 - k + \frac{p^2}{N^2} \ell(\ell + 1) \right) P_\ell(y), \quad \ell = 0, \dots, p - 1. \quad (5.15)$$

It is then clear that $b_{n,\ell}$ in (4.25) is non zero only for $n = \ell$ if $\ell < p$. For application in (4.46) we may note that the contribution of $\frac{1}{2}$ -BPS multiplets is given by

$$(-1)^j (b_{j,j} + \gamma_{j-1,1} b_{j-1,j-1}) = \frac{j!^2}{(2j-1)!} \frac{p^2}{N^2}, \quad j = 1, \dots, p - 1. \quad (5.16)$$

In general as a consequence of (4.36) and (4.37) $\{b_{n,\ell}\}$ determines for which cases the index I_{nm} defined in (1.4) is non zero.

We here adapt the discussion in the previous section to determine the free field form for $\hat{\mathcal{H}}_t$, which was defined in (4.41) and is responsible for the contributions of long multiplets. First with the definition (4.33) it is convenient to further define, following (4.34),

$$\begin{aligned} \mathcal{F}_t^{(j)} &= \delta_{t,j} \mathcal{Q}_j^0 + \delta_{t,j+1} (\mathcal{Q}_j^1 + \gamma_{j,-1} \mathcal{Q}_{j-1}^0) \\ &\quad + \delta_{t,j+2} (\gamma_{j,-1} \mathcal{Q}_{j-1}^1 + c_j \mathcal{Q}_j^0) + \delta_{t,j+3} \gamma_{j,-1} c_{j+1} \mathcal{Q}_{j-1}^0. \end{aligned} \quad (5.17)$$

This satisfies

$$\mathcal{F}_t^{(j)}(x, \sigma, \tau) + \gamma_{j,1} \mathcal{F}_t^{(j+1)}(x, \sigma, \tau) = \sum_j c_{t,j} \mathcal{Q}_j^0(x, \sigma, \tau), \quad (5.18)$$

which is essentially equivalent to (4.38). Starting from (4.15) with (4.19) and (4.26) then

$$\begin{aligned} \mathcal{G}_t^{(p)}(x, \sigma, \tau) - k \mathcal{G}_t(\mathcal{I}; x) &= \sum_{\ell \geq 0} c_{t,\ell} \frac{(\alpha - \frac{1}{x}) f_\ell(\alpha) - (\bar{\alpha} - \frac{1}{x}) f_\ell(\bar{\alpha})}{\alpha - \bar{\alpha}} \\ &= \left(\alpha - \frac{1}{x} \right) \left(\bar{\alpha} - \frac{1}{x} \right) \left(\sum_j c_{t,j} \mathcal{H}_j(x, \sigma, \tau) + \mathcal{F}_t^{(0)}(x, \sigma, \tau) \right) \\ &= \left(\alpha - \frac{1}{x} \right) \left(\bar{\alpha} - \frac{1}{x} \right) \left(\sum_j c_{t,j} \hat{\mathcal{H}}_j(x, \sigma, \tau) + (-1)^J \frac{J!^2}{(2J)!} \mathcal{F}_t^{(J)}(x, \sigma, \tau) \right). \end{aligned} \quad (5.19)$$

where in this case we have by using (5.18)

$$\hat{\mathcal{H}}_j(x, \sigma, \tau) = \mathcal{H}_j(x, \sigma, \tau) + (-1)^j \frac{j!^2}{(2j)!} \mathcal{Q}_j^0(x, \sigma, \tau), \quad j = 0, 1, \dots, J-1. \quad (5.20)$$

This definition of $\hat{\mathcal{H}}_j$ is essentially equivalent to (4.41), using (5.20) the expansion $\hat{\mathcal{H}}_j(x, \sigma, \tau) = \sum_{\ell} A_{j\ell}(\sigma, \tau) g_{1,j+\ell+2}(x)$ determines the corresponding expansion for \mathcal{H}_j given by (4.41) with coefficients $\frac{1}{2}(A_{j\ell} - A_{j+\ell+1} - \ell - 2)$, manifestly satisfying (3.4). The result (5.19) is similarly equivalent to (4.42). Following previous considerations we take $J = p$.

The result (5.19) may be simplified by noting that

$$\sum_{\ell \geq 0} c_{t,\ell} (-1)^{\ell} \frac{\ell!^2}{(2\ell)!} \frac{(\alpha - \frac{1}{x})P_{\ell}(y) - (\bar{\alpha} - \frac{1}{x})P_{\ell}(\bar{y})}{\alpha - \bar{\alpha}} = \mathcal{G}_t(\mathcal{I}; x), \quad (5.21)$$

since then the $1 - k$ terms in (5.15) are just sufficient to change the coefficient of $\mathcal{G}_t(\mathcal{I}; x)$ in (5.19) to 1. From (5.12) we have

$$\mathcal{G}_t^{(p)}(x, \sigma, \tau) = \mathcal{G}_t(\mathcal{I}; x) + \frac{p^2}{N^2} \tilde{\mathcal{G}}_t(x, \sigma, \tau), \quad t < p, \quad (5.22)$$

and hence (5.19), with (5.15), gives

$$\begin{aligned} \tilde{\mathcal{G}}_t(x, \sigma, \tau) &= \sum_{\ell > 0} c_{t,\ell} (-1)^{\ell} \frac{\ell!^2}{(2\ell)!} \ell(\ell+1) \frac{(\alpha - \frac{1}{x})P_{\ell}(y) - (\bar{\alpha} - \frac{1}{x})P_{\ell}(\bar{y})}{\alpha - \bar{\alpha}} \\ &\quad + \left(\alpha - \frac{1}{x}\right) \left(\bar{\alpha} - \frac{1}{x}\right) \sum_j c_{t,j} \mathcal{K}_j(x, \sigma, \tau), \end{aligned} \quad (5.23)$$

where

$$\hat{\mathcal{H}}_j(x, \sigma, \tau) = \frac{p^2}{N^2} \mathcal{K}_j(x, \sigma, \tau), \quad j < p. \quad (5.24)$$

It is clear that \mathcal{K}_j is universal, independent of the particular p .

The left hand side of (5.23) is determined by the expansion

$$\left(\frac{1}{x} - \frac{1}{\bar{x}}\right) \tilde{\mathcal{G}}(u, v; \sigma, \tau) = \sum_t \tilde{\mathcal{G}}_t(x, \sigma, \tau) g_{1,t-1}(\bar{x}). \quad (5.25)$$

Writing from (5.13)

$$\begin{aligned} \left(\frac{1}{x} - \frac{1}{\bar{x}}\right) \tilde{\mathcal{G}}(u, v; \sigma, \tau) &= -\sigma x + \tau x' + \sigma \frac{1 - \sigma x + \tau x'}{1 - \sigma x - \tau x'} (1 - \sigma x^2) \frac{\bar{x}}{1 - \sigma x \bar{x}} \\ &\quad + \tau \frac{1 + \sigma x - \tau x'}{1 - \sigma x - \tau x'} (1 - \tau x'^2) \frac{\bar{x}}{1 - (1 - \tau x')\bar{x}}, \end{aligned} \quad (5.26)$$

we have

$$\tilde{\mathcal{G}}_1(x, \sigma, \tau) = -\sigma x + \tau x', \quad (5.27)$$

and also, using (3.12), then $\tilde{\mathcal{G}}_t$ is determined for $t = 2, 3, \dots$ by

$$\begin{aligned} \tilde{\mathcal{G}}_{t+2}(x, \sigma, \tau) = & -\frac{t!^2}{(2t)!} \left(\sigma \frac{1 - \sigma x + \tau x'}{1 - \sigma x - \tau x'} (1 - \sigma x^2) P_t(1 - 2\sigma x) \right. \\ & \left. + \tau \frac{1 + \sigma x - \tau x'}{1 - \sigma x - \tau x'} (1 - \tau x'^2) P_t(2\tau x' - 1) \right). \end{aligned} \quad (5.28)$$

It is easy to see that this satisfies (4.13). For application in (5.23) we consider modified expansion

$$\left(\frac{1}{x} - \frac{1}{\bar{x}} \right) \tilde{\mathcal{G}}(u, v; \sigma, \tau) = \sum_{t,j} c_{t,j} \tilde{\mathcal{G}}'_j(x, \sigma, \tau) g_{1,t-1}(\bar{x}). \quad (5.29)$$

Using (4.20) this is equivalent to

$$\frac{\bar{x}^2}{(1 - \alpha \bar{x})(1 - \bar{\alpha} \bar{x})} \left(\frac{1}{x} - \frac{1}{\bar{x}} \right) \tilde{\mathcal{G}}(u, v; \sigma, \tau) = \sum_j \tilde{\mathcal{G}}'_j(x, \sigma, \tau) g_{1,j+1}(\bar{x}). \quad (5.30)$$

By decomposing into partial fractions of the form (3.12) we obtain

$$\begin{aligned} \tilde{\mathcal{G}}'_j(x, \sigma, \tau) = & -(-1)^j \frac{j!^2}{(2j)!} \frac{(1-x)^2}{(1-\alpha x)^2(1-\bar{\alpha} x)^2} \\ & \times \left(b P_j(2\sigma x - 1) + c P_j(1 - 2\tau x') + \frac{a P_j(y) - \bar{a} P_j(\bar{y})}{\alpha - \bar{\alpha}} \right), \\ b = & \frac{(1 - \sigma x^2)(1 - \sigma x + \tau x')}{1 - x}, \quad c = (1 - x)(1 - \tau x'^2)(1 + \sigma x - \tau x'), \\ a = & \frac{(1 - \alpha x)^3(\bar{y} - \bar{\alpha} x)}{(1 - x)^2}, \quad \bar{a} = \frac{(1 - \bar{\alpha} x)^3(y - \alpha x)}{(1 - x)^2}. \end{aligned} \quad (5.31)$$

This then gives

$$\mathcal{K}_j(x, \sigma, \tau) = -(-1)^j \frac{j!^2}{(2j)!} \left(B P_j(2\sigma x - 1) + C P_j(1 - 2\tau x') + \frac{A_j P_j(y) - \bar{A}_j P_j(\bar{y})}{\alpha - \bar{\alpha}} \right), \quad (5.32)$$

with

$$\begin{aligned} B = & -2\bar{\alpha} x^2 \frac{(1 - \alpha)(1 - \bar{\alpha})}{(1 - \bar{\alpha} x)^3(\alpha - \bar{\alpha})^3} (\alpha - \bar{\alpha}^2 x) - \frac{\bar{\alpha} x^2}{(1 - \bar{\alpha} x)^2(\alpha - \bar{\alpha})} + \alpha \leftrightarrow \bar{\alpha}, \\ A_j = & \frac{x^2(\bar{y} - \bar{\alpha} x)}{(1 - \bar{\alpha} x)^3} - j(j+1) \frac{x}{1 - \bar{\alpha} x}, \quad \bar{A}_j = A_j|_{\alpha \leftrightarrow \bar{\alpha}}, \quad C = B|_{\alpha \rightarrow 1-\alpha, \bar{\alpha} \rightarrow 1-\bar{\alpha}, x \rightarrow x'}. \end{aligned} \quad (5.33)$$

In general the results given by (5.32) with (5.33) are not straightforward to analyse. However for the leading term for large σ, τ we have

$$\mathcal{K}_j(x, \sigma, \tau) \sim (-1)^j \left((\sigma(\sigma x)^j - \tau(-\tau x')^j - (\sigma - \tau)^{j+1}) \frac{\sigma x - \tau x'}{(\sigma x + \tau x')^3} x^2 x'^2 - j(j+1)(\sigma - \tau)^j \frac{x x'}{\sigma x + \tau x'} \right). \quad (5.34)$$

Hence writing in general

$$\mathcal{K}_j(x, \sigma, \tau) = (-1)^j \sum_{g, h \geq 0} \mathcal{K}_{gh,j}(x) \sigma^g \tau^h, \quad \mathcal{K}_{gh,j}(x) = (-1)^j \mathcal{K}_{hg,j}(x'), \quad (5.35)$$

then $\mathcal{K}_{gh,j} = 0$ for $g + h \geq j$ and from expanding (5.34) so that the denominators $\sigma x + \tau x'$ are cancelled,

$$\begin{aligned} \mathcal{K}_{gh, g+h+1}(x) = (-1)^h & \left((h+1)^2 x^{g+1} x'^{h+2} - x'^2 \sum_{r=0}^h \binom{g+h+2}{r} (h+1-r)^2 \left(\frac{x'}{x}\right)^{h-r} \right. \\ & \left. - x'(g+h+1)(g+h+2) \sum_{r=0}^h \binom{g+h+1}{r} \left(\frac{x'}{x}\right)^{h-r} \right). \end{aligned} \quad (5.36)$$

Using $x^n x'^m = \frac{1}{(m-1)!} \sum_{r=1}^n \frac{(n+m-r-1)!}{(n-r)!} x^r + \frac{1}{(n-1)!} \sum_{r=1}^m \frac{(n+m-r-1)!}{(m-r)!} x'^r$ for $n, m \geq 1$ and $x'/x = x' - 1$ this may be written as a polynomial in x of degree $g+1$ and in x' of degree $h+1$. For general j there is no apparent general formula for $\mathcal{K}_{gh,j}(x)$ but we note that

$$\mathcal{K}_{00,j}(x) = \frac{j!^2}{(2j)!} j(j+1)(x + (-1)^j x'), \quad (5.37)$$

and we have also obtained $\mathcal{K}_{10,j}(x)$ in accord with results in appendix C.

6. Expressions for General Chiral Four Point Functions

In this section we suggest general expressions for the dynamical part of the large N amplitude for the four point function (2.1) for identical single trace $\frac{1}{2}$ -BPS operators belonging to the $SU(4)$ $[0, p, 0]$ representation. This will be based on what has been observed in the examples for $p = 2, 3, 4$ which have been explicitly calculated using the AdS/CFT correspondence [14, 18, 19]. For $p \geq 4$ there are also multi-trace BPS operators but our considerations do not apply for these.

As described in section 4 the dynamical part of the four point function, for general p , reduces to $\mathcal{H}^{(p)}$, polynomial of degree $p-2$ in σ, τ , which as a consequence of (2.4) satisfies the crossing relations

$$\mathcal{H}^{(p)}(u, v; \sigma, \tau) = \frac{1}{v^2} \mathcal{H}^{(p)}(u/v, 1/v; \tau, \sigma) = \left(\frac{u}{v}\right)^{p-2} \tau^{p-2} \mathcal{H}^{(p)}(v, u; \sigma/\tau, 1/\tau). \quad (6.1)$$

The results for $p = 2, 3, 4$ can be reduced to the following form

$$\mathcal{H}^{(p)}(u, v; \sigma, \tau) = -\frac{p^2}{N^2} u^p \sum_{\substack{0 \leq b \leq a \\ 2a+b \leq p-2}} \sum_{i,j,k} c_{ijk,ab}^{(p)} T_{ijk,ab}^{(p)}(u, v; \sigma, \tau), \quad (6.2)$$

where $T_{ijk,ab}^{(p)}$ are completely crossing symmetric combinations of \overline{D} functions which are related to the crossing symmetric polynomials in (5.4). For $p = 2, 3, \dots$ and restricting $0 \leq b \leq a$, $2a + b \leq p - 2$, corresponding to (5.5), we define

$$\begin{aligned} T_{ijk,ab}^{(p)}(u, v; \sigma, \tau) &= \frac{1}{6} n_{ab}^{(p-2)} (\sigma^a \tau^b \overline{D}_{ip+2jk}(u, v) + \sigma^b \tau^a \overline{D}_{ip+2kj}(u, v) \\ &\quad + \sigma^a \tau^{p-2-a-b} \overline{D}_{jp+2ik}(u, v) + \sigma^{p-2-a-b} \tau^a \overline{D}_{jp+2ki}(u, v) \\ &\quad + \sigma^b \tau^{p-2-a-b} \overline{D}_{kp+2ij}(u, v) + \sigma^{p-2-a-b} \tau^b \overline{D}_{kp+2ji}(u, v)). \end{aligned} \quad (6.3)$$

The crossing identities for \overline{D} functions ensure that $u^p T_{ijk,ab}^{(p)}(u, v; \sigma, \tau)$ satisfies (6.1) without any additional factors of u, v being necessary. The factors $\frac{1}{6} n_{ab}^{(p-2)}$ are introduced for later convenience, essentially since, for the boundary values of a, b , we have

$$\begin{aligned} T_{ijk,aa}^{(p)} &= T_{ikj,aa}^{(p)}, \quad T_{ijk,a\,p-2-2a}^{(p)} = T_{kji,a\,p-2-2a}^{(p)} = T_{jki,aa}^{(p)}, \\ T_{ijk,\frac{1}{3}(p-2)\frac{1}{3}(p-2)}^{(p)} &= T_{(ijk),\frac{1}{3}(p-2)\frac{1}{3}(p-2)}^{(p)}. \end{aligned} \quad (6.4)$$

When a, b satisfy the conditions in (6.4) then (6.3) can be simplified in appropriate cases,

$$\begin{aligned} T_{ijj,aa}^{(p)}(u, v; \sigma, \tau) &= T_{jij,a\,p-2-2a}^{(p)}(u, v; \sigma, \tau) \\ &= \sigma^a \tau^a \overline{D}_{ip+2jj}(u, v) + \sigma^a \tau^{p-2-2a} \overline{D}_{jp+2ij}(u, v) + \sigma^{p-2-2a} \tau^a \overline{D}_{jp+2ji}(u, v), \\ T_{iii,\frac{1}{3}(p-2)\frac{1}{3}(p-2)}^{(p)}(u, v; \sigma, \tau) &= (\sigma\tau)^{\frac{1}{3}(p-2)} \overline{D}_{ip+2ii}(u, v). \end{aligned} \quad (6.5)$$

Further restrictions arise from the unitarity conditions for each long multiplet which contribute in the operator product expansion. For the lowest scale dimension operator belonging to the representation $[n - m, 2m, n - m]$ and with spin ℓ we must have $\Delta \geq 2n + \ell + 2$. To apply this if we expand

$$\mathcal{H}^{(p)}(u, v; \sigma, \tau) = \sum_{0 \leq m \leq n \leq p-2} A_{nm}(u, v) Y_{nm}(\sigma, \tau), \quad (6.6)$$

where $Y_{nm}(\sigma, \tau)$ are the polynomials in (2.8), then the unitarity condition requires

$$A_{nm}(u, v) = O(u^{n+1}). \quad (6.7)$$

The \overline{D} functions that can appear in (6.3) are constrained by the unitarity conditions (6.7). To obtain these we first list the essential properties used here, see [12,18]. For $\overline{D}_{n_1 n_2 n_3 n_4}(u, v)$ we define

$$\Sigma = \frac{1}{2}(n_1 + n_2 + n_3 + n_4), \quad s = \frac{1}{2}(n_1 + n_2 - n_3 - n_4), \quad (6.8)$$

and, restricting to Σ an integer, for $s = 1, 2, \dots$ we have

$$\begin{aligned} \overline{D}_{n_1 n_2 n_3 n_4}(u, v) = & \sum_{m=0}^{s-1} u^{-s+m} \frac{(-1)^m}{m!} (s-m-1)! f_{n_1-s+m, n_2-s+m, n_3+m, n_4+m}(v) \\ & + \ln u a_{n_1 n_2 n_3 n_4}(u, v) + b_{n_1 n_2 n_3 n_4}(u, v), \end{aligned} \quad (6.9)$$

where $a_{n_1 n_2 n_3 n_4}(u, v)$, $b_{n_1 n_2 n_3 n_4}(u, v)$ are both given by power series in $u, 1-v$. For $s = 0$ the first term in (6.9) is omitted. In (6.9) the relevant terms here are given in terms of ordinary hypergeometric functions

$$f_{n_1 n_2 n_3 n_4}(v) = \frac{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_4)}{\Gamma(n_3 + n_4)} F(n_2, n_3; n_3 + n_4; 1-v). \quad (6.10)$$

From standard relations for hypergeometric functions we have

$$n_1 f_{n_1 n_2 n_3 n_4}(v) = f_{n_1+1, n_2, n_3+1, n_4}(v) + f_{n_1+1, n_2, n_3, n_4+1}(v), \quad (6.11)$$

and also

$$f_{n_1 n_2 n_3 n_4}(v) = v^{-n_2} f_{n_1 n_2 n_4 n_3}(1/v). \quad (6.12)$$

The $\ln u$ terms in (6.9) result in anomalous dimensions for operators belonging to long multiplets which have twist $\Delta - \ell \geq 2p$ at zeroth order in the $1/N$ expansion, where Δ is the scale dimension and ℓ the spin. The terms involving negative powers u^{-s+m} have no corresponding $\ln u$ terms and would correspond in the operator product expansion to contributions from long multiplets which are unrenormalised. We assume that these are all cancelled although there remain contributions from various semi-short multiplets which cannot be combined to form a long multiplet, thus all multiplets not satisfying a shortening condition gain anomalous dimensions in the $1/N$ expansion as expected. This is the essential assumption that leads to strong constraints on the expansion (6.2).

Using the properties of the $SU(4)$ harmonics $Y_{nm}(\sigma, \tau)$ it is clear that the unitarity condition (6.7) is satisfied if for any contribution in (6.2) involving

$$\sigma^g \tau^h u^p \overline{D}_{n_1, p+2, n_3, n_4}(u, v), \quad (6.13)$$

we require

$$0 \leq s \leq p - g - h - 1. \quad (6.14)$$

With the assumed expansion given by (6.2) and (6.3) the condition $s \geq 0$ ensures from (6.9) that only long multiplets with twist $\Delta - \ell \geq 2p$ with non zero anomalous dimensions in the large N limit can contribute to the operator product expansion of the chiral four point function.

Applying the condition (6.14) to (6.3) gives the following inequalities

$$\begin{aligned} p - 2a &\leq i + j - k \leq p + 2, \\ p - 2b &\leq i + k - j \leq p + 2, \\ 2(a + b + 2) - p &\leq j + k - i \leq p + 2. \end{aligned} \tag{6.15}$$

We also impose

$$i, j, k \leq p. \tag{6.16}$$

It is clear that there is a finite number of possibilities for i, j, k , note that $p+4 \leq i+j+k \leq 3p$. We should also note that the expansion (6.3) is not unique since

$$\frac{1}{2}(i + j + k - p - 2)\overline{D}_{i\,p+2\,j\,k+1} = \overline{D}_{i\,p+2\,j+1\,k+1} + \overline{D}_{i+1\,p+2\,j\,k+1} + \overline{D}_{i+1\,p+2\,j+1\,k}. \tag{6.17}$$

This is the only relation for \overline{D} functions of the form appearing in (6.3). Correspondingly we may take

$$\frac{1}{2}(i + j + k - p - 2)T_{ijk,ab}^{(p)} = T_{ij+1\,k+1,ab}^{(p)} + T_{i+1\,j\,k+1,ab}^{(p)} + T_{i+1\,j+1\,k,ab}^{(p)}, \tag{6.18}$$

which allows the expansion (6.2) to be simplified if each term in (6.18) satisfies the constraints (6.15) and (6.16). For this to be the case we must have

$$i + j - k, i + k - j, j + k - i \leq p, \quad i, j, k \leq p - 1. \tag{6.19}$$

Whenever (6.19) is satisfied one of the terms appearing in (6.18) may be omitted in the general expansion.

If we use (6.19) to remove terms with the lowest value of $i+j+k$ whenever appropriate then for general p the list of possible terms obtained from (6.15), (6.16) modulo (6.19) is

$$\begin{aligned} &T_{pjj,ab}^{(p)}, \quad j = a + b + 2, \dots, p, \\ &T_{iip,ab}^{(p)}, \quad i = p - a, \dots, p - 1, \quad a \geq 1, \\ &T_{ipi,ab}^{(p)}, \quad i = p - b, \dots, p - 1, \quad b \geq 1, \\ &T_{ii+k-p-2\,k,ab}^{(p)}, \quad i = p - a + 1, \dots, p, \quad k = a + b + 3, \dots, p, \quad a \geq 1, \\ &T_{ij\,i+j-p-2,ab}^{(p)}, \quad i = p - b + 1, \dots, p, \quad j = a + b + 3, \dots, p, \quad b \geq 1, \\ &T_{j+k-p-2\,jk,ab}^{(p)}, \quad j = p - a + 1, \dots, p, \quad k = p - b + 1, \dots, p, \quad a, b \geq 1. \end{aligned} \tag{6.20}$$

When $a = b$ or $b = p - 2 - 2a$ it is necessary also to take into account the symmetry conditions in (6.4) to obtain an independent basis. If $N_{ab}^{(p)}$ are the number of possibilities for each a, b then we have

$$\begin{aligned} N_{ab}^{(p)} &= (a + b + 1)(p - a - b - 1) + ab, \quad 0 \leq b < a < \frac{1}{2}(p - 2 - b), \\ N_{aa}^{(p)} &= (a + 1)(p - \frac{3}{2}a - 1), \quad 0 \leq a < \frac{1}{3}(p - 2), \\ N_{a, p-2-2a}^{(p)} &= (a + 1)(p - \frac{3}{2}a - 1), \quad \frac{1}{3}(p - 2) < a \leq \frac{1}{2}(p - 2), \\ N_{\frac{1}{3}(p-2), \frac{1}{3}(p-2)}^{(p)} &= \frac{1}{18}(p + 1)(p + 4), \quad p = 2 \bmod 3. \end{aligned} \quad (6.21)$$

The crucial assumption, in addition to (6.3), is that the leading terms in the expansion of $\mathcal{H}^{(p)}$ in powers of u , which do not involve any $\ln u$ terms, are universal, i.e. we have for any $p = 2, 3, \dots$

$$\mathcal{H}^{(p)}(u, v; \sigma, \tau) = -\frac{p^2}{N^2} \mathcal{F}(u, v; \sigma, \tau) + O(u^p), \quad (6.22)$$

where \mathcal{F} is independent of p . From (6.1) we have

$$\mathcal{F}(u, v; \sigma, \tau) = \frac{1}{v^2} \mathcal{F}(u/v, 1/v; \tau, \sigma), \quad (6.23)$$

and it is expressible in general as an expansion

$$\mathcal{F}(u, v; \sigma, \tau) = \sum_{n \geq g+h+2} \sum_{g, h \geq 0} \sigma^g \tau^h u^{n-1} \mathcal{F}_{n, gh}(v). \quad (6.24)$$

With the expansion (6.2) for $\mathcal{H}^{(p)}$, and with $T_{ijk, ab}^{(p)}$ given by (6.3), then only the leading singular terms in the expansion of the \overline{D} functions shown in (6.9) contribute to \mathcal{F} in (6.22). Assuming $T_{ijk, ab}^{(p)}$ are restricted as in (6.20) then the potential contributions to $\mathcal{F}_{n, gh}$ in (6.24) which are given by (6.22) are just

$$\begin{aligned} f_{n-1, n+1, nn}, \quad & n = g + h + 2, \dots, p, \\ f_{j-1, n+1, nj}, \quad & j = n - h, \dots, n, \quad n = p, \\ f_{j-1, n+1, jn}, \quad & j = n - g, \dots, n, \quad n = p, \\ f_{j, n+1, j, n+1}, \quad & j = n - g, \dots, n - 1, \quad n = g + h + 2, \dots, p, \\ f_{j, n+1, n+1, j}, \quad & j = n - h, \dots, n - 1, \quad n = g + h + 2, \dots, p, \end{aligned} \quad (6.25)$$

which may be further reduced with the aid of (6.11).

A additional restriction, which is compatible with the results for $p = 2, 3, 4$, is that only the terms with $n = g + h + 2$ are present in (6.24) so that

$$\begin{aligned} \mathcal{F}(u, v; \sigma, \tau) &= \sum_{g, h \geq 0} \sigma^g \tau^h u^{g+h+1} \mathcal{F}_{gh}(v) \\ \mathcal{F}_{gh}(v) &= \sum_{l, m} d_{gh, lm} f_{l+m-g-h-3, g+h+3, lm}(v), \end{aligned} \quad (6.26)$$

where l, m are restricted in accord with (6.25) with $n = g + h + 2$. Apart from the relation (6.11) the functions $u^n f_{l+m-n-2n+2lm}(v)$ appearing in (6.26) are assumed to be linearly independent. With the restricted form in (6.26) then, as shown later, we are both able to determine \mathcal{F} from (6.22) and using the explicit form of \mathcal{F} , up to terms of $O(u^{p-1})$, to find all the coefficients $c_{ijk,ab}^{(p)}$ which appear in the expansion (6.2).

A simple consequence of (6.22) and (6.2), which does not require any restrictions of the form for \mathcal{F} , is that for $p = 3, 4, \dots$,

$$\begin{aligned} u^p \sum_{\substack{0 \leq b \leq a \\ 2a+b \leq p-2}} \sum_{i,j,k} c_{ijk,ab}^{(p)} T_{ijk,ab}^{(p)}(u, v; \sigma, \tau) \\ = u^{p-1} \sum_{\substack{0 \leq b \leq a \\ 2a+b \leq p-3}} \sum_{i,j,k} c_{ijk,ab}^{(p-1)} T_{ijk,ab}^{(p-1)}(u, v; \sigma, \tau) + O(u^{p-1}), \end{aligned} \quad (6.27)$$

where only the leading singular terms displayed explicitly in (6.9) need be considered. These equations are invariant under $\sigma \leftrightarrow \tau$ and $u \rightarrow u/v, v \rightarrow 1/v$. In (6.27) all \overline{D} functions with $s = 1, 2, \dots$ are relevant on the right hand side but only those with $s = 2, 3, \dots$ on the left hand side. It is important to note that (6.27) does not constrain, $c_{ppp,ab}^{(p)}$, the coefficient for \overline{D}_{pp+2pp} which is present for any a, b .

7. Applications for Low p

We now show how the above suggestions work out in practice for low p , initially using only (6.27).

For $p = 2$ $\mathcal{H}^{(2)}$ is independent of σ, τ and there is just one possible \overline{D} function which is in accord with the simplification of results obtained from AdS/CFT [14,23],

$$\mathcal{H}^{(2)}(u, v) = -\frac{4}{N^2} u^2 \overline{D}_{2422}(u, v). \quad (7.1)$$

For $p = 3$ we must have again $a = b = 0$ and there are just two crossing symmetric forms, [18],

$$\begin{aligned} \mathcal{H}^{(3)}(u, v; \sigma, \tau) &= -\frac{9}{N^2} u^3 \left(c_{322,00}^{(3)} T_{322,00}^{(3)} + c_{333,00}^{(3)} T_{333,00}^{(3)} \right), \\ &= -\frac{9}{N^2} u^3 \left(c_{333,00}^{(3)} (1 + \sigma + \tau) \overline{D}_{3533} + c_{322,00}^{(3)} (\overline{D}_{3522} + \sigma \overline{D}_{2523} + \tau \overline{D}_{2532}) \right). \end{aligned} \quad (7.2)$$

The equations (6.27) give just one condition arising from $u^3 \overline{D}_{3522}(u, v) \sim u f_{1322}(v)$ and in (7.1) $u^2 \overline{D}_{2422}(u, v) \sim u f_{1322}(v)$ so that we require

$$c_{322,00}^{(3)} = 1. \quad (7.3)$$

The known results also give

$$c_{333,00}^{(3)} = 1. \quad (7.4)$$

For $p = 4$, for comparison with previous results, we write

$$\begin{aligned} \mathcal{H}^{(4)}(u, v; \sigma, \tau) = & -\frac{16}{N^2} u^4 \left(c_{422,00}^{(4)} T_{422,00}^{(4)} + c_{433,00}^{(4)} T_{433,00}^{(4)} + c_{444,00}^{(4)} T_{444,00}^{(4)} \right. \\ & \left. + c_{323,10}^{(4)} T_{323,10}^{(4)} + c_{424,10}^{(4)} T_{424,10}^{(4)} + c_{444,10}^{(4)} T_{444,10}^{(4)} \right), \end{aligned} \quad (7.5)$$

where from (6.4) and (6.18) $T_{334,10}^{(4)} = T_{433,10}^{(4)} = \frac{1}{2}(T_{323,10}^{(4)} - T_{424,10}^{(4)})$ so that such contributions are discarded. Expanding (7.5) then gives

$$\begin{aligned} \mathcal{H}^{(4)}(u, v; \sigma, \tau) = & -\frac{16}{N^2} u^4 \left((c_{444,00}^{(4)}(1 + \sigma^2 + \tau^2) + c_{444,10}^{(4)}(\sigma + \tau + \sigma\tau)) \bar{D}_{4644} \right. \\ & + c_{323,10}^{(4)}(\sigma \bar{D}_{3623} + \tau \bar{D}_{3632} + \sigma\tau \bar{D}_{2633}) \\ & + c_{424,10}^{(4)}(\sigma \bar{D}_{4624} + \tau \bar{D}_{4642} + \sigma\tau \bar{D}_{2644}) \\ & + c_{422,00}^{(4)}(\bar{D}_{4622} + \sigma^2 \bar{D}_{2624} + \tau^2 \bar{D}_{2642}) \\ & \left. + c_{433,00}^{(4)}(\bar{D}_{4633} + \sigma^2 \bar{D}_{3634} + \tau^2 \bar{D}_{3643}) \right). \end{aligned} \quad (7.6)$$

Applying (6.27) then gives for the 1 terms

$$c_{422,00}^{(4)} = \frac{1}{2}, \quad c_{422,00}^{(4)} - c_{433,00}^{(4)} = 0, \quad (7.7)$$

and from the σ terms

$$c_{323,10}^{(4)} f_{1423}(v) + c_{424,10}^{(4)} f_{2424}(v) = f_{2433}(v) + f_{1423}(v). \quad (7.8)$$

Using (6.11) this is easily solved giving

$$c_{323,10}^{(4)} = 2, \quad c_{424,10}^{(4)} = -1. \quad (7.9)$$

The remaining coefficients which are undetermined by (6.27) are

$$c_{444,10}^{(4)} = 1, \quad c_{444,00}^{(4)} = \frac{1}{4}. \quad (7.10)$$

For the basis corresponding to (6.20) then, instead of (7.6), we should take

$$\begin{aligned} \mathcal{H}^{(4)}(u, v; \sigma, \tau) = & -\frac{16}{N^2} u^4 \left(\sum_{j=2}^4 c_{4jj,00}^{(4)} T_{4jj,00}^{(4)} \right. \\ & \left. + 2c_{433,10}^{(4)} T_{433,10}^{(4)} + c_{424,10}^{(4)} T_{424,10}^{(4)} + c_{444,10}^{(4)} T_{444,10}^{(4)} \right). \end{aligned} \quad (7.11)$$

Here we introduce a factor 2 for the $c_{433,10}^{(4)}$ terms to count equal contributions from $T_{433,10}^{(4)}$ and $T_{343,10}^{(4)}$. This ensures uniformity with later general results. In this case (7.7) and (7.9) are unchanged but instead of (7.9) we should take

$$c_{433,10}^{(4)} = 2, \quad c_{424,10}^{(4)} = 1. \quad (7.12)$$

On the basis of the results for $p = 2, 3, 4$ we determine the first few terms in the function \mathcal{F} introduced in (6.22),

$$\begin{aligned} \mathcal{F}(u, v; \sigma, \tau) = & u f_{1322}(v) \\ & + \sigma u^2 (f_{1423}(v) + f_{2433}(v)) + \tau u^2 (f_{1432}(v) + f_{2433}(v)) \\ & + \sigma^2 u^3 \frac{1}{2} (f_{1524}(v) + f_{2534}(v) + \frac{1}{2} f_{3544}(v)) \\ & + \tau^2 u^3 \frac{1}{2} (f_{1542}(v) + f_{2543}(v) + \frac{1}{2} f_{3544}(v)) \\ & + \sigma \tau u^3 (2f_{1533}(v) + f_{3544}(v)) + \mathcal{O}(u^4). \end{aligned} \quad (7.13)$$

This result is in accord with the assumed form in (6.26).

For $p = 5$ we have the general form

$$\begin{aligned} \mathcal{H}^{(5)}(u, v; \sigma, \tau) = & -\frac{25}{N^2} u^5 \left(\sum_{j=2}^5 c_{5jj,00}^{(5)} T_{5jj,00}^{(5)} \right. \\ & + \sum_{j=3}^5 c_{5jj,10}^{(5)} T_{5jj,10}^{(5)} + c_{445,10}^{(5)} T_{445,10}^{(5)} \\ & + c_{524,10}^{(5)} T_{524,10}^{(5)} + c_{535,10}^{(5)} T_{535,10}^{(5)} \\ & \left. + 3c_{544,11}^{(5)} T_{544,11}^{(5)} + 3c_{553,11}^{(5)} T_{553,11}^{(5)} + c_{555,11}^{(5)} T_{555,11}^{(5)} \right). \end{aligned} \quad (7.14)$$

Here we note from (6.4) that $T_{ijk,11}^{(5)}$ is completely symmetric in i, j, k and $\frac{1}{3} T_{333,11}^{(5)} = T_{443,11}^{(5)} = T_{544,11}^{(5)} + \frac{1}{2} T_{553,11}^{(5)}$, so neither of these terms are present in the expansion in (7.14). By using (6.18), we also eliminate terms involving $T_{434,10}^{(5)}$, $T_{423,10}^{(5)}$. As in (7.11) we introduce factors to take account of the sum over identical terms related by (6.4). With the basis in (7.14) and the results for $\mathcal{H}^{(4)}$ we may readily solve the equations (6.27) giving

$$\begin{aligned} c_{522,00}^{(5)} = c_{533,00}^{(5)} &= \frac{1}{6}, & c_{544,00}^{(5)} &= \frac{1}{12}, \\ c_{533,10}^{(5)} = c_{544,10}^{(5)} &= 1, & c_{524,10}^{(5)} = c_{535,10}^{(5)} &= \frac{1}{2}, & c_{445,10}^{(5)} &= \frac{3}{4}, \\ c_{544,11}^{(5)} &= 3, & c_{553,11}^{(5)} &= 1. \end{aligned} \quad (7.15)$$

Only $c_{555,00}^{(5)}$, $c_{555,10}^{(5)}$, $c_{555,11}^{(5)}$ are undetermined at this stage. In the expansion (6.24) we may

now obtain

$$\begin{aligned}
\sum_{gh} \sigma^g \tau^h \mathcal{F}_{5,gh}(v) &= (c_{555,00}^{(5)} - \frac{1}{36}) f_{4655}(v) \\
&+ (\sigma + \tau + \sigma^2 + \tau^2) (c_{555,10}^{(5)} - \frac{1}{4}) f_{4655}(v) \\
&+ \sigma \tau (c_{555,11}^{(5)} - 1) f_{4655}(v) \\
&+ \sigma^3 (\frac{1}{6} f_{1625}(v) + \frac{1}{6} f_{2635}(v) + \frac{1}{12} f_{3645}(v) + c_{555,00}^{(5)} f_{4655}(v)) \\
&+ \tau^3 (\frac{1}{6} f_{1652}(v) + \frac{1}{6} f_{2653}(v) + \frac{1}{12} f_{3654}(v) + c_{555,00}^{(5)} f_{4655}(v)) \\
&+ \sigma^2 \tau (f_{1634}(v) + \frac{1}{2} f_{2644}(v) + \frac{1}{4} f_{3645}(v) + c_{555,10}^{(5)} f_{4655}(v)) \\
&+ \sigma \tau^2 (f_{1643}(v) + \frac{1}{2} f_{2644}(v) + \frac{1}{4} f_{3654}(v) + c_{555,10}^{(5)} f_{4655}(v)). \tag{7.16}
\end{aligned}$$

The restrictions, imposed by the additional constraint that just the leading term for $n = g + h + 1$ appears in the expansion of \mathcal{F} for each g, h as assumed in (6.26), is easily achieved in (7.16) by setting

$$c_{555,00}^{(5)} = \frac{1}{36}, \quad c_{555,10}^{(5)} = \frac{1}{4}, \quad c_{555,11}^{(5)} = 1. \tag{7.17}$$

For $p = 6$ we have the following expansion in terms of independent crossing symmetric functions, noting that $T_{ijk,11}^{(6)} = T_{ikj,11}^{(6)}$ and $T_{635,20}^{(6)} = T_{536,20}^{(6)}$,

$$\begin{aligned}
\mathcal{H}^{(6)}(u, v; \sigma, \tau) &= -\frac{36}{N^2} u^6 \left(\sum_{j=2}^6 c_{6jj,00}^{(6)} T_{6jj,00}^{(6)} \right. \\
&+ \sum_{j=3}^6 c_{6jj,10}^{(6)} T_{6jj,10}^{(5)} + c_{556,10}^{(6)} T_{556,10}^{(6)} \\
&+ c_{624,10}^{(6)} T_{624,10}^{(6)} + c_{635,10}^{(6)} T_{635,10}^{(6)} + c_{636,10}^{(6)} T_{636,10}^{(6)} \\
&+ \sum_{j=4}^6 c_{6jj,11}^{(6)} T_{6jj,11}^{(6)} + 2c_{556,11}^{(6)} T_{556,11}^{(6)} \\
&+ 2c_{635,11}^{(6)} T_{635,11}^{(6)} + 2c_{646,11}^{(6)} T_{646,11}^{(6)} + c_{466,11}^{(6)} T_{466,11}^{(6)} \\
&\left. + 2c_{635,20}^{(6)} T_{635,20}^{(6)} + c_{646,20}^{(6)} T_{646,20}^{(6)} + c_{525,20}^{(6)} T_{525,20}^{(6)} \right). \tag{7.18}
\end{aligned}$$

Here for $a = b = 1$ and $a = 2, b = 0$ we have used (6.4) to reduce the number of necessary terms. In this case the equations give

$$\begin{aligned}
c_{622,00}^{(6)} &= c_{633,00}^{(6)} = \frac{1}{24}, \quad c_{644,00}^{(6)} = \frac{1}{48}, \quad c_{655,00}^{(6)} = -\frac{1}{48} + c_{555,00}^{(5)}, \\
c_{633,10}^{(6)} &= c_{644,10}^{(6)} = \frac{1}{3}, \quad c_{655,10}^{(6)} = -\frac{1}{12} + c_{555,10}^{(5)}, \\
c_{624,10}^{(6)} &= c_{635,10}^{(6)} = \frac{1}{6}, \quad c_{646,10}^{(6)} = \frac{1}{12}, \quad c_{556,10}^{(6)} = \frac{1}{12} + c_{555,00}^{(5)}, \\
c_{644,11}^{(6)} &= \frac{3}{2}, \quad c_{655,11}^{(6)} = \frac{1}{2} + c_{555,11}^{(5)}, \quad c_{466,11}^{(6)} = \frac{1}{4}, \\
c_{635,11}^{(6)} &= c_{646,11}^{(6)} = \frac{1}{2}, \quad c_{556,11}^{(6)} = \frac{3}{4} + c_{555,10}^{(5)}, \\
c_{644,20}^{(6)} &= \frac{3}{8}, \quad c_{655,20}^{(6)} = \frac{1}{8} + c_{555,10}^{(5)}, \quad c_{525,20}^{(6)} = c_{646,20}^{(6)} = \frac{1}{4}, \quad c_{635,20}^{(6)} = \frac{1}{4}. \tag{7.19}
\end{aligned}$$

We may now extend the results given by (7.13) and (7.16), assuming (7.17), to obtain

$$\begin{aligned}
\sum_{gh} \sigma^g \tau^h \mathcal{F}_{6,gh}(v) = & (c_{666,00}^{(6)} - \frac{1}{576}) f_{5766}(v) \\
& + (\sigma + \tau + \sigma^3 + \tau^3) (c_{666,10}^{(6)} - \frac{1}{36}) f_{5766}(v) \\
& + \sigma \tau (1 + \sigma + \tau) (c_{666,11}^{(6)} - \frac{1}{4}) f_{5766}(v) \\
& + (\sigma^2 + \tau^2) (c_{666,20}^{(6)} - \frac{1}{16}) f_{5766}(v) \\
& + \sigma^4 \left(\frac{1}{24} f_{1726}(v) + \frac{1}{24} f_{2736}(v) + \frac{1}{48} f_{3746}(v) + \frac{1}{144} f_{4756}(v) + c_{666,00}^{(6)} f_{5766}(v) \right) \\
& + \tau^4 \left(\frac{1}{24} f_{1762}(v) + \frac{1}{24} f_{2763}(v) + \frac{1}{48} f_{3764}(v) + \frac{1}{144} f_{4765}(v) + c_{666,00}^{(6)} f_{5766}(v) \right) \\
& + \sigma^3 \tau \left(\frac{1}{3} f_{1735}(v) + \frac{1}{3} f_{2745}(v) + \frac{1}{18} f_{4756}(v) + c_{666,10}^{(6)} f_{5766}(v) \right) \\
& + \sigma \tau^3 \left(\frac{1}{3} f_{1753}(v) + \frac{1}{3} f_{2754}(v) + \frac{1}{18} f_{4765}(v) + c_{666,10}^{(6)} f_{5766}(v) \right) \\
& + \sigma^2 \tau^2 \left(\frac{3}{4} f_{1744}(v) + \frac{3}{8} f_{3755}(v) + c_{666,20}^{(6)} f_{5766}(v) \right). \tag{7.20}
\end{aligned}$$

Again the same fashion as (7.17) and in accord with (6.26) we also obtain

$$c_{666,00}^{(6)} = \frac{1}{576}, \quad c_{666,10}^{(6)} = \frac{1}{36}, \quad c_{666,11}^{(6)} = \frac{1}{4}, \quad c_{666,20}^{(6)} = \frac{1}{16}. \tag{7.21}$$

8. General Solutions

We here discuss how the equations which follow from (6.22), assuming (6.2) with $T_{ijk,ab}^{(p)}$ restricted as in (6.20), can be solved if we also suppose that the only contributions in the expansion for \mathcal{F} are restricted to the form shown in (6.26). The general expansion has the form

$$\begin{aligned}
\mathcal{H}^{(p)} = & -\frac{p^2}{N^2} u^p \sum_{\substack{0 \leq b \leq a \\ 2a+b \leq p-2}} \left(\sum_{j=a+b+2}^p c_{pjj,ab}^{(p)} T_{pjj,ab}^{(p)} + \sum_{i=p-a}^{p-1} c_{iip,ab}^{(p)} T_{iip,ab}^{(p)} + \sum_{i=p-b}^{p-1} c_{ipi,ab}^{(p)} T_{ipi,ab}^{(p)} \right. \\
& + \sum_{i=p-a+1}^p \sum_{k=a+b+3}^p c_{i i+k-p-2 k,ab}^{(p)} T_{i i+k-p-2 k,ab}^{(p)} \\
& + \sum_{i=p-b+1}^p \sum_{j=a+b+3}^p c_{ij i+j-p-2,ab}^{(p)} T_{ij i+j-p-2,ab}^{(p)} \\
& \left. + \sum_{j=p-a+1}^p \sum_{k=p-b+1}^p c_{j+k-p-2 jk,ab}^{(p)} T_{j+k-p-2 jk,ab}^{(p)} \right). \tag{8.1}
\end{aligned}$$

We first consider terms independent of σ, τ which arise only from $\sum_{j=2}^p c_{pjj,00}^{(p)} T_{pjj,00}^{(p)}$ where $T_{pjj,00}^{(p)} \rightarrow \overline{D}_{pp+2jj}$. Requiring $\mathcal{F}(u, v; \sigma, \tau) = u f_{1322}(v) + O(\sigma, \tau)$ as in (7.13) we

obtain

$$\sum_{j=2}^p \sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p-j-1)! c_{pjj,00}^{(p)} u^{j+m-1} f_{j-1+m \ j+1+m \ j+m \ j+m}(v) = u f_{1322}(v), \quad (8.2)$$

which requires

$$(p-2)! c_{p22,00}^{(p)} = 1, \quad \sum_{m=0}^{k-2} \frac{(-1)^m}{m!} c_{p \ k-m \ k-m,00}^{(p)} = 0, \quad k = 3, \dots, p. \quad (8.3)$$

This is easily solved giving

$$c_{pjj,00}^{(p)} = \frac{1}{(p-2)! (j-2)!}, \quad j = 2, \dots, p. \quad (8.4)$$

We next consider the calculation of the coefficients $c_{ijk,ab}^{(p)}$ for $a \geq 1, b = 0$. These are determined in (6.22) by the terms in the expansion (6.26) with $g = a, h = 0$. Contributions proportional to σ^g first arise in an expansion in powers of u of $\mathcal{H}^{(p)}$ from $\sum_{j=2}^p c_{pjj,00}^{(p)} T_{pjj,00}^{(p)}$, with $T_{pjj,00}^{(p)}(u, v; \sigma, \tau) \rightarrow \sigma^p \bar{D}_{j \ p+2 \ jp}(u, v)$, for $p = g + 2$. Using (6.22) with (8.4) this gives the relevant contribution to the expansion of \mathcal{F} in (6.26) for this case,

$$\mathcal{F}_{g0}(v) = \frac{1}{g!} \sum_{j=2}^{g+2} \frac{1}{(j-2)!} f_{j-1 \ g+3 \ j \ g+2}(v). \quad (8.5)$$

Assuming this form for \mathcal{F} in general then for $p \geq a+3$ (6.22) requires, for the contributions which involve powers u^n with $n < p$ arising only from the $T_{pjj,a0}^{(p)}, T_{i \ i+k-p-2 \ k,a0}^{(p)}$ and $T_{iip,a0}^{(p)}$ terms in (8.1), keeping just the first term in (6.3),

$$\begin{aligned} & \sum_{j=a+2}^p c_{pjj,a0}^{(p)} \sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p-j-m)! u^{j+m-1} f_{j-1+m \ j+1+m \ j+m \ j+m}(v) \\ & + \sum_{i=p-a+1}^p \sum_{k=a+3}^p c_{i \ i+k-p-2 \ k,a0}^{(p)} \sum_{m=0}^{p-k+1} \frac{(-1)^m}{m!} (p-k+1-m)! \\ & \quad \times u^{k+m-2} f_{i+k-p-2+m \ k+m \ i+k-p-2+m \ k+m}(v) \\ & + \sum_{i=p-a}^{p-1} c_{iip,a0}^{(p)} u^{p-1} f_{i-1 \ p+1 \ ip}(v) \\ & = u^{a+1} \frac{1}{a!} \sum_{j=2}^{a+2} \frac{1}{(j-2)!} f_{j-1 \ a+3 \ j \ a+2}(v). \end{aligned} \quad (8.6)$$

To analyse (8.6) we consider first all terms proportional to u^{a+1} when we obtain

$$\begin{aligned}
& (p-a-2)! c_{p a+2 a+2, a0}^{(p)} f_{a+1 a+3 a+2 a+2}(v) \\
& + (p-a-2)! \sum_{i=p-a+1}^p c_{i i+a-p+1 a+3, a0}^{(p)} f_{i+a-p+1 a+3 i+a-p+1 a+3}(v) \\
& = \frac{1}{a!} \sum_{j=2}^{a+2} \frac{1}{(j-2)!} f_{j-1 a+3 j a+2}(v).
\end{aligned} \tag{8.7}$$

Applying (6.11) repeatedly for $f_{j-1 a+3 j a+2}(v)$ we then get

$$\begin{aligned}
c_{i i+a-p+1 a+3, a0}^{(p)} &= \frac{1}{(p-a-2)! a! (i+a-p-1)!}, \quad i = p-a+1, \dots, p, \\
c_{p a+2 a+2, a0}^{(p)} &= \frac{a+1}{(p-a-2)! a!^2}.
\end{aligned} \tag{8.8}$$

From contributions in (8.6) proportional to $u^{k-1} f_{k-1 k+1 k k}(v)$, for $k = a+3, \dots, p-1$, and $u^{k-2} f_{i+k-p-2 k i+k-p-2 k}(v)$, for $k = a+4, \dots, p$, $i = p-a+1, \dots, p$, we get

$$\sum_{m=0}^{k-a-2} \frac{(-1)^m}{m!} c_{p k-m k-m, a0}^{(p)} = 0, \quad \sum_{m=0}^{k-a-3} \frac{(-1)^m}{m!} c_{i i+k-p-2-m k-m, a0}^{(p)} = 0, \tag{8.9}$$

which in conjunction with (8.8) may be solved giving

$$\begin{aligned}
c_{p j j, a0}^{(p)} &= \frac{a+1}{(p-a-2)! a!^2 (j-a-2)!}, \quad j = a+2, \dots, p-1, \\
c_{i i+k-p-2 k, a0}^{(p)} &= \frac{1}{(p-a-2)! a! (i+a-p-1)! (k-a-3)!}, \\
&\quad k = a+3, \dots, p, \quad i = p-a+1, \dots, p.
\end{aligned} \tag{8.10}$$

Using (8.10) the remaining part of (8.6) becomes

$$\begin{aligned}
& \left(c_{p p p, a0}^{(p)} - \frac{a+1}{(p-a-2)!^2 a!^2} \right) f_{p-1 p+1 p p}(v) \\
& - \frac{1}{(p-a-2)!^2 a!} \sum_{j=p-a+1}^p \frac{1}{(j+a-p-1)!} f_{j-1 p+1 j-1 p+1}(v) \\
& + \sum_{i=p-a}^{p-1} c_{i i p, a0}^{(p)} f_{i-1 p+1 i p}(v) = 0.
\end{aligned} \tag{8.11}$$

With the aid of (6.11) again we finally obtain for this case

$$\begin{aligned}
c_{p p p, a0}^{(p)} &= \frac{1}{(p-a-2)!^2 a!^2}, \\
c_{i i p, a0}^{(p)} &= \frac{p-a-1}{(p-a-2)!^2 a! (i+a-p)!}, \quad i = p-a, \dots, p-1.
\end{aligned} \tag{8.12}$$

For $p = 2a + 2$ the symmetry conditions (6.4) ensure that in the expansion (8.1) we may require $c_{pjj,a0}^{(p)} = c_{jjp,a0}^{(p)}$ and $c_{i+k-p-2,k,a0}^{(p)}$ is symmetric in i, k . With these constraints instead of (8.6) we have now

$$\begin{aligned}
& \sum_{j=a+2}^p c_{pjj,a0}^{(p)} \left(\sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p-j-m)! u^{j+m-1} f_{j-1+m} j_{+1+m} j_{+m} j_{+m}(v) \right. \\
& \quad \left. + u^{p-1} f_{j-1} p_{+1} j_p(v) \right) \\
& + \sum_{i,k=a+3}^p c_{i+k-p-2,k,a0}^{(p)} \sum_{m=0}^{p-k+1} \frac{(-1)^m}{m!} (p-k+1-m)! \\
& \quad \times u^{k+m-2} f_{i+k-p-2+m} k_{+m} i_{+k-p-2+m} k_{+m}(v) \\
& = u^{a+1} \frac{1}{a!} \sum_{j=2}^{a+2} \frac{1}{(j-2)!} f_{j-1} a_{+3} j_{a+2}(v).
\end{aligned} \tag{8.13}$$

The solution of (8.13) is essentially as before giving in this case

$$\begin{aligned}
c_{ppp,a0}^{(p)} &= \frac{1}{a!^4}, \quad c_{pjj,a0}^{(p)} = \frac{a+1}{a!^3 (j-a-2)!}, \quad j = a+2, \dots, p-1. \\
c_{i+k-p-2,k,a0}^{(p)} &= \frac{1}{a!^2 (i-a-3)! (k-a-3)!}, \quad i, k = a+3, \dots, p.
\end{aligned} \tag{8.14}$$

These results are just as expected from (8.10) and (8.12) after substituting $p = 2a + 2$. The results manifestly satisfy the necessary symmetry conditions.

For completeness it is also necessary to analyse the other contributions which are present in crossing symmetric expressions exhibited in (6.3) for $T_{pjj,a0}^{(p)}$, $T_{i+k-p-2,k,a0}^{(p)}$ and $T_{iip,a0}^{(p)}$. The six terms in (6.3) form three pairs related by $\sigma \leftrightarrow \tau$ under which our equations are invariant. From those terms proportional to σ^{p-2-a} we obtain

$$\begin{aligned}
& \sum_{j=a+2}^p c_{pjj,a0}^{(p)} u^{p-1} f_{j-1} p_{+1} j_p(v) \\
& + \sum_{i=p-a+1}^p \sum_{k=a+3}^p c_{i+k-p-2,k,a0}^{(p)} \sum_{m=0}^{p-i+1} \frac{(-1)^m}{m!} (p-i+1-m)! \\
& \quad \times u^{i+m-2} f_{i+k-p-2+m} i_{+m} i_{+k-p-2+m} i_{+m}(v) \\
& + \sum_{i=p-a}^{p-1} c_{iip,a0}^{(p)} \sum_{m=0}^{p-i} \frac{(-1)^m}{m!} (p+1-i-m)! u^{i+m-1} f_{i-1+m} i_{+1+m} i_{+m} i_{+m}(v) \\
& = u^{p-1-a} \frac{1}{(p-2-a)!} \sum_{j=2}^{p-a} \frac{1}{(j-2)!} f_{j-1} p_{-a+1} j_{p-a}(v) \\
& = u^{p-1-a} \mathcal{F}_{p-2-a,0}(v),
\end{aligned} \tag{8.15}$$

using (8.5). The identity shown in (8.15) is obtained by following the identical procedure as in calculations described above after using

$$c_{i i+k-p-2 k, a 0}^{(p)} \leftrightarrow c_{k i+k-p-2 i, a 0}^{(p)}, \quad c_{p j j, a 0}^{(p)} \leftrightarrow c_{j j p, a 0}^{(p)}, \quad \text{for } a \rightarrow p-a-2, \quad (8.16)$$

which are easily seen to be a property of the solutions (8.10) and (8.12). The result (8.15) is then in accord with expectation from (6.22).

For the remaining terms we consider those proportional to $\sigma^{p-2-a}\tau^a$ for which, up to terms of $O(u^p)$, we have just

$$\sigma^{p-2-a}\tau^a u^{p-1} \left(\sum_{j=a+2}^p c_{p j j, a 0}^{(p)} f_{j-1 p+1 j p}(v) + \sum_{i=p-a}^{p-1} c_{i i p, a 0}^{(p)} f_{i-1 p+1 p i}(v) \right). \quad (8.17)$$

These are identified as required by (6.22) with the following term in the expansion (6.26), after using the expressions (8.10) and (8.12),

$$\begin{aligned} \mathcal{F}_{gh}(v) = & \sum_{j=h+2}^{g+h+1} \frac{h+1}{g! h!^2 (j-h-2)!} f_{j-1 g+h+3 j g+h+2}(v) \\ & + \sum_{j=g+2}^{g+h+1} \frac{g+1}{g!^2 h! (j-g-2)!} f_{j-1 g+h+3 g+h+2 j}(v) \\ & + \frac{1}{g!^2 h!^2} f_{g+h+1 g+h+3 g+h+2 g+h+2}(v). \end{aligned} \quad (8.18)$$

This result is obtained from (8.17) for $g > h$, the corresponding result for $g < h$ is obtained by using the symmetry under $\sigma \leftrightarrow \tau$, for $g = h$ it is necessary to use (8.16). For $h = 0$ (8.18) coincides with (8.5). Although (8.18) is not immediately of the form expected from (6.26) and (6.25), it can be reduced to it by application of (6.11).

With the determination of \mathcal{F} in general in (8.18) we can now determine the remaining coefficients in (8.1). For terms proportional to $\sigma^a \tau^b$ in (6.22) we have, corresponding to

(8.6),

$$\begin{aligned}
& \sum_{j=a+b+2}^p c_{pjj,ab}^{(p)} \sum_{m=0}^{p-j} \frac{(-1)^m}{m!} (p-j-m)! u^{j+m-1} f_{j-1+m} j+1+m j+m j+m(v) \\
& + \sum_{i=p-a+1}^p \sum_{k=a+b+3}^p c_{i i+k-p-2 k,ab}^{(p)} \sum_{m=0}^{p-k+1} \frac{(-1)^m}{m!} (p-k+1-m)! \\
& \quad \times u^{k+m-2} f_{i+k-p-2+m} k+m i+k-p-2+m k+m(v) \\
& + \sum_{i=p-b+1}^p \sum_{j=a+b+3}^p c_{i j i+j-p-2,ab}^{(p)} \sum_{m=0}^{p-j+1} \frac{(-1)^m}{m!} (p-j+1-m)! \\
& \quad \times u^{j+m-2} f_{i+j-p-2+m} j+m j+m i+j-p-2+m(v) \\
& + \sum_{i=p-a}^{p-1} c_{iip,ab}^{(p)} u^{p-1} f_{i-1} p+1 i p(v) + \sum_{i=p-b}^{p-1} c_{ipi,ab}^{(p)} u^{p-1} f_{i-1} p+1 p i(v) \\
& = u^{a+b+1} \left(\sum_{j=b+2}^{a+b+1} \frac{b+1}{a! b!^2 (j-b-2)!} f_{j-1} a+b+3 j a+b+2(v) \right. \\
& \quad + \sum_{j=a+2}^{a+b+1} \frac{a+1}{a!^2 b! (j-a-2)!} f_{j-1} a+b+3 a+b+2 j(v) \\
& \quad \left. + \frac{1}{a!^2 b!^2} f_{a+b+1} a+b+3 a+b+2 a+b+2(v) \right). \tag{8.19}
\end{aligned}$$

This may be analysed in a similar fashion to previously. For terms proportional to u^{a+b+1} ,

$$\begin{aligned}
& (p-a-b-2)! c_{p a+b+2 a+b+2,ab}^{(p)} f_{a+b+1} a+b+3 a+b+2 a+b+2(v) \\
& + (p-a-b-2)! \sum_{i=p-a+1}^p c_{i i+a+b-p+1 a+b+3,ab}^{(p)} f_{i+a+b-p+1} a+b+3 i+a+b-p+1 a+b+3(v) \\
& + (p-a-b-2)! \sum_{i=p-b+1}^p c_{i a+b+3 i+a+b-p+1,ab}^{(p)} f_{i+a+b-p+1} a+b+3 a+b+3 i+a+b-p+1(v) \\
& = \sum_{j=b+2}^{a+b+1} \frac{b+1}{a! b!^2 (j-b-2)!} f_{j-1} a+b+3 j a+b+2(v) \\
& + \sum_{j=a+2}^{a+b+1} \frac{a+1}{a!^2 b! (j-a-2)!} f_{j-1} a+b+3 a+b+2 j(v) + \frac{1}{a!^2 b!^2} f_{a+b+1} a+b+3 a+b+2 a+b+2(v). \tag{8.20}
\end{aligned}$$

Using (6.11) this may be decomposed to give

$$c_{i a+b+i-p+1 a+b+3,ab}^{(p)} = \frac{1}{(p-a-b-2)! a! b!^2 (i+a-p-1)!}, \quad i = p-a+1, \dots, p,$$

$$\begin{aligned}
c_{i a+b+3 a+b+i-p+1, ab}^{(p)} &= \frac{1}{(p-a-b-2)! a!^2 b! (i+b-p-1)!}, \quad i = p-b+1, \dots, p, \\
c_{p a+b+2 a+b+2, ab}^{(p)} &= \frac{a+b+1}{(p-a-b-2)! a!^2 b!^2}.
\end{aligned} \tag{8.21}$$

To obtain (8.21) we have used the identity $\sum_{m=0}^k (n-1+m)!/m! = (n+k)!/n k!$. For terms proportional to u^n in (8.19) for $n = a+b+2, \dots, p-2$ we get

$$\begin{aligned}
\sum_{m=0}^{k-a-b-2} \frac{(-1)^m}{m!} c_{p k-m k-m, ab}^{(p)} &= 0, \quad k = a+b+3, \dots, p-1, \\
\sum_{m=0}^{k-a-b-3} \frac{(-1)^m}{m!} c_{i i+k-p-2-m k-m, ab}^{(p)} &= 0, \quad \begin{cases} i = p-a+1, \dots, p \\ k = a+b+4, \dots, p \end{cases}, \\
\sum_{m=0}^{j-a-b-3} \frac{(-1)^m}{m!} c_{i j-m i+j-p-2-m, ab}^{(p)} &= 0, \quad \begin{cases} i = p-b+1, \dots, p \\ j = a+b+4, \dots, p \end{cases}.
\end{aligned} \tag{8.22}$$

Combining with (8.21) then gives

$$\begin{aligned}
c_{p j j, ab}^{(p)} &= \frac{a+b+1}{(p-a-b-2)! a!^2 b!^2 (j-a-b-2)!}, \quad j = a+b+2, \dots, p-1, \\
c_{i i+k-p-2 k, ab}^{(p)} &= \frac{1}{(p-a-b-2)! a! b!^2 (i+a-p-1)! (k-a-b-3)!}, \\
&\quad k = a+b+3, \dots, p, \quad i = p-a+1, \dots, p, \\
c_{i j i+j-p-2, ab}^{(p)} &= \frac{1}{(p-a-b-2)! a!^2 b! (i+b-p-1)! (j-a-b-3)!}, \\
&\quad j = a+b+3, \dots, p, \quad i = p-b+1, \dots, p.
\end{aligned} \tag{8.23}$$

For the remaining terms in (8.19) proportional to u^{p-1} after using the result (8.23) we have

$$\begin{aligned}
&\left(c_{p p p, ab}^{(p)} - \frac{a+b+1}{(p-a-b-2)! a!^2 b!^2} \right) f_{p-1 p+1 p p}(v) \\
&- \frac{1}{(p-a-b-2)!^2 a! b!^2} \sum_{j=p-a+1}^p \frac{1}{(j+a-p-1)!} f_{j-1 p+1 j-1 p+1}(v) \\
&- \frac{1}{(p-a-b-2)!^2 a!^2 b!} \sum_{j=p-b+1}^p \frac{1}{(j+b-p-1)!} f_{j-1 p+1 p+1 j-1}(v) \\
&+ \sum_{i=p-a}^{p-1} c_{i i p, ab}^{(p)} f_{i-1 p+1 i p}(v) + \sum_{i=p-b}^{p-1} c_{i p i, ab}^{(p)} f_{i-1 p+1 p i}(v) = 0.
\end{aligned} \tag{8.24}$$

This may be solved giving

$$\begin{aligned}
c_{ppp,ab}^{(p)} &= \frac{1}{(p-a-b-2)!^2 a!^2 b!^2}, \\
c_{iip,ab}^{(p)} &= \frac{p-a-1}{(p-a-b-2)!^2 a! b!^2 (i+a-p)!}, \quad i = p-a, \dots, p-1, \\
c_{ipi,ab}^{(p)} &= \frac{p-b-1}{(p-a-b-2)!^2 a!^2 b! (i+b-p)!}, \quad i = p-b, \dots, p-1.
\end{aligned} \tag{8.25}$$

The coefficients satisfy the crucial relations

$$\begin{aligned}
c_{iij+p-2j,ab}^{(p)} &\leftrightarrow c_{jij+p-2i,ab}^{(p)}, & c_{ijij+p-2,ab}^{(p)} &\leftrightarrow c_{iij+p-2j,ab}^{(p)}, \\
c_{iip,ab}^{(p)} &\leftrightarrow c_{pii,ab}^{(p)}, & c_{ipi,ab}^{(p)} &\leftrightarrow c_{ip i,ab}^{(p)}, \quad \text{for } a \rightarrow p-2-a-b,
\end{aligned} \tag{8.26}$$

and

$$\begin{aligned}
c_{iij+p-2j,ab}^{(p)} &\leftrightarrow c_{iij+p-2ij,ab}^{(p)}, & c_{ijij+p-2,ab}^{(p)} &\leftrightarrow c_{jij+p-2,ab}^{(p)}, \\
c_{iip,ab}^{(p)} &\leftrightarrow c_{iip,ab}^{(p)}, & c_{ipi,ab}^{(p)} &\leftrightarrow c_{pii,ab}^{(p)}, \quad \text{for } b \rightarrow p-2-a-b.
\end{aligned} \tag{8.27}$$

There is also a similar relation for $a \leftrightarrow b$ which can be obtained by combining (8.26) and (8.27). These relations require for the undetermined coefficient so far

$$c_{j+k-p-2jk,ab}^{(p)} = \frac{1}{(p-a-b-2)!^2 a! b! (j+a-p-1)! (k+b-p-1)!}, \tag{8.28}$$

$j = p-a+1, \dots, p, k = p-b+1, \dots, p.$

The results (8.23), (8.25) and (8.28) hence determine the expansion of $\mathcal{H}^{(p)}$ for general p . The symmetry conditions (8.26) and (8.27) are necessary to ensure that the other terms in the expression for $T_{ijk,ab}^{(p)}$, defined in (6.3), contribute as required to the terms in \mathcal{F} proportional to $\sigma^g \tau^h$ with $g = p-2-a-b$, $h = b$ and $g = a$, $h = p-2-a-b$. The results given by (8.23), (8.25) and (8.28) also satisfy

$$\begin{aligned}
c_{iij+p-2,aa}^{(p)} &= c_{iij+p-2j,aa}^{(p)}, & c_{j+k-p-2jk,aa}^{(p)} &= c_{j+k-p-2kj,aa}^{(p)}, \\
c_{iip,aa}^{(p)} &= c_{ipi,aa}^{(p)}, \\
c_{ijij+p-2,a\,p-2-2a}^{(p)} &= c_{iij+p-2ji,a\,p-2-2a}^{(p)}, & c_{ij+k-p-2k,a\,p-2-2a}^{(p)} &= c_{kij+k-p-2i,a\,p-2-2a}^{(p)}, \\
c_{iip,a\,p-2-2a}^{(p)} &= c_{pii,a\,p-2-2a}^{(p)},
\end{aligned} \tag{8.29}$$

which shows that they remain valid in these cases when the symmetry requirements in (6.4) hold.

9. Cancellation with Free Field Results

We here endeavour to show that the universal function $\mathcal{F}(u, v; \sigma, \tau)$, defined by (6.22) and given by (6.26) with (8.18), are precisely what is required to cancel the universal free field results which have an expansion of the form (5.35). To demonstrate this we need to consider an expansion in terms of contributions of differing twist,

$$(x - \bar{x}) u^{g+h+2} \mathcal{F}_{gh}(v) = -(-1)^j \sum_{j \geq g+h+1} \mathcal{F}_{gh,j}(x) g_{1,j+1}(\bar{x}). \quad (9.1)$$

The leading term is easily seen to be

$$\mathcal{F}_{gh,g+h+1}(x) = x^{g+h+3} \mathcal{F}_{gh}(1-x). \quad (9.2)$$

To obtain an appropriate expression $\mathcal{F}_{gh,j}(x)$ we follow the basic strategy outlined in section three by using an integral representation for the hypergeometric functions in term of which the basic functions $f_{n_1 n_2 n_3 n_4}$, as defined in (6.11), are expressed. Considering those which appear in the solution (5.35) for \mathcal{F} we obtain, for $n = 1, \dots, N$,

$$\begin{aligned} & x^{N+2} f_{n N+2 n+1 N+1}(1-x) \\ &= -(n-1)! N! \left(x' + (-1)^n \sum_{s=0}^{N-n} \frac{(N-s)! s!}{(N-n-s)! (s+n)!} x^{s+1} \right) + \sum_{\ell=0}^{n-1} r_{\ell}^{(n,N)} g_{1,\ell+1}(x), \\ & (-1)^N x^{N+2} f_{n N+2 N+1 n+1}(1-x) \\ &= -(n-1)! N! \left(x + (-1)^n \sum_{s=0}^{N-n} \frac{(N-s)! s!}{(N-n-s)! (s+n)!} x'^{s+1} \right) - \sum_{\ell=0}^{n-1} (-1)^{\ell} r_{\ell}^{(n,N)} g_{1,\ell+1}(x). \end{aligned} \quad (9.3)$$

A proof is given in appendix C where an expression for $r_{\ell}^{(n,N)}$ is obtained.

Using (9.3) with (9.2) and (8.18) gives

$$\begin{aligned} \mathcal{F}_{gh,g+h+1}(x) &= (-1)^h \left(\frac{h+1}{h!} \sum_{s=0}^g \frac{(g+h+1-s)!}{(g-s)!} x^{s+1} - \frac{g+1}{g!} \sum_{s=0}^h \frac{(g+h+1-s)!}{(h-s)!} x'^{s+1} \right) \\ &\quad + \sum_{\ell=0}^{g+h} s_{\ell} g_{1,\ell+1}(x), \end{aligned} \quad (9.4)$$

with s_{ℓ} a linear combination of $r_{\ell}^{(n,g+h+3)}$. The first term in (9.4) is identical with $\mathcal{K}_{gh,g+h+1}(x)$ as given by (5.36) after rewriting in terms of polynomials in x and x' . This demonstrates the cancellation of the leading term of $\mathcal{F}(u, v; \sigma, \tau)$ in a twist expansion with the corresponding terms found in free field theory.

The analysis for $\mathcal{F}_{gh,j}(x)$ for $j > g + h + 1$ is more involved. Here we consider just $\mathcal{F}_{00}(v) = f_{1322}(v)$. As shown in appendix C the expansion (9.1) gives for this case

$$\mathcal{F}_{00,j}(x) = \frac{j!^2}{(2j)!} j(j+1)(x + (-1)^j x') + \frac{j!^2}{(2j)!} \sum_{\ell=0}^{j-1} \frac{\ell!^2}{(2\ell)!} c_{j,\ell} g_{1,\ell+1}(x), \quad (9.5)$$

where

$$c_{j,\ell} = (j - \ell)(j + \ell + 1)(1 - (-1)^{j-\ell}). \quad (9.6)$$

Up to a finite number of partial waves this is identical with $\mathcal{K}_{00,j}(x)$ as given by (5.37) which demonstrates the cancellation in this case. The method used to obtain (9.5) is generalisable to other cases but we have not extended it in general because of algebraic complexity.

10. Conclusion

We have shown how expressions for four point functions for identical chiral primary $\frac{1}{2}$ -BPS operators may be obtained for arbitrary p , albeit we should note that we require $p \ll N$. Direct calculations from supergravity assuming the AdS/CFT correspondence are highly non trivial although our results, if correct, may suggest that calculations for any p should be possible. The techniques used here of avoiding $SU(4)$ complications by reducing all expressions to polynomials may be of assistance. At order $1/N^2$ the supergravity calculations must reproduce the function $f(x, \alpha)$, for which there are no perturbative corrections and which is therefore given by free field theory, as well as the dynamical part contained in $\mathcal{H}(u, v; \sigma, \tau)$. The cancellations between these contributions suggest that this separation, although following from the superconformal Ward identities, is less natural from the viewpoint of supergravity.

It is also possible that the approach described here might also be applicable for four point functions involving differing $\frac{1}{2}$ -BPS operators, although the constraints of crossing symmetry which played a crucial role would no longer be present. If feasible this reflects the remarkable properties of $\mathcal{N} = 4$ superconformal theories in the large N limit.

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Appendix A. Results for Short and Semi-Short Multiplets

We here illustrate the results of section four by working out more fully the contributions of a short and semi-short multiplets. Taking $f(x, \alpha) = \frac{1}{2}g_{1,j+k+2}(x)P_i(y)$ then (4.32) and (4.33) give

$$\begin{aligned}
\mathcal{G}_{\mathcal{Q}_j^0,t}^{(j)} &= (\delta_{t,j} \sum_n c_{n,j+k+1} g_{1,n-1} - c_{t,j+k+1} g_{1,j-1}) Y_{j-1,i}, \\
\mathcal{G}_{\mathcal{Q}_j^1,t}^{(j+1)} &= (\delta_{t,j+1} \sum_n c_{n,j+k+1} g_{1,n-1} - c_{t,j+k+1} g_{1,j}) (\gamma_{i,1} Y_{j-1,i+1} + \gamma_{i,-1} Y_{j-1,i-1}), \\
\mathcal{G}_{\mathcal{Q}_{j-1}^0,t}^{(j+1)} &= (\delta_{t,j+1} \sum_n c_{n,j+k+1} g_{1,n-1} - c_{t,j+k+1} g_{1,j}) Y_{j-2,i}, \\
\mathcal{G}_{\mathcal{Q}_{j-1}^1,t}^{(j+2)} &= (\delta_{t,j+2} \sum_n c_{n,j+k+1} g_{1,n-1} - c_{t,j+k+1} g_{1,j+1}) (\gamma_{i,1} Y_{j-2,i+1} + \gamma_{i,-1} Y_{j-2,i-1}), \\
\mathcal{G}_{\mathcal{Q}_j^0,t}^{(j+2)} &= (\delta_{t,j+2} \sum_n c_{n,j+k+1} g_{1,n-1} - c_{t,j+k+1} g_{1,j+1}) Y_{j-1,i}, \\
\mathcal{G}_{\mathcal{Q}_{j-1}^0,t}^{(j+3)} &= (\delta_{t,j+3} \sum_n c_{n,j+k+1} g_{1,n-1} - c_{t,j+k+1} g_{1,j+2}) Y_{j-2,i}.
\end{aligned} \tag{A.1}$$

Hence the definitions (4.34) and (4.36) lead to

$$\mathcal{G}_{j+r}(\mathcal{C}_{j-1,i,k}) = \sum_{s=0}^3 a_{r,s} g_{1,j+k+s}, \quad \mathcal{G}_{j+k+1+r}(\mathcal{C}_{j-1,i,k}) = -\sum_{s=0}^3 a_{s,r} g_{1,j-1+s}, \tag{A.2}$$

for $r = 0, 1, 2, 3$ and where

$$\begin{aligned}
a_{0,s} &= c_{j+k+s+1,j+k+1} Y_{j-1,i}, \\
a_{1,s} &= c_{j+k+s+1,j+k+1} (\gamma_{i,1} Y_{j-1,i+1} + \gamma_{i,-1} Y_{j-1,i-1} + \gamma_{j,-1} Y_{j-2,i}), \\
a_{2,s} &= c_{j+k+s+1,j+k+1} (\gamma_{j,-1} (\gamma_{i,1} Y_{j-2,i+1} + \gamma_{i,-1} Y_{j-2,i-1}) + c_j Y_{j-1,i}), \\
a_{3,s} &= c_{j+k+s+1,j+k+1} \gamma_{j,-1} c_{j+1} Y_{j-2,i}.
\end{aligned} \tag{A.3}$$

The contribution of a semi-short multiplet in the conformal partial wave expansion is then determined by

$$\sum_{nm} a_{nm,j+r,k+s-r}(\mathcal{C}_{j-1,i,k}) Y_{nm} = a_{r,s}, \tag{A.4}$$

which is easily calculated from (4.18) and (2.10).

From (4.37) and (A.2) for $j > i$ we have setting $k = -1$

$$\mathcal{G}_{j+r}(\mathcal{B}_{ji}) = \sum_{s=r+1}^4 b_{r,s} g_{1,j+s-1} - \sum_{s=0}^{r-1} b_{s,r} g_{1,j+s-1}, \quad \gamma_{j,1} b_{r,s} = a_{r,s} - a_{s,r}, \tag{A.5}$$

and hence from (4.34) and using (2.9), the coefficients are given by

$$\begin{aligned}
b_{0,1} &= Y_{ji} , \\
b_{0,2} &= \gamma_{i,1} Y_{j\ i+1} + \gamma_{i,-1} Y_{j\ i-1} + \gamma_{j+1,-1} Y_{j-1\ i} , \\
b_{0,3} &= c_{j+1} Y_{ji} + \gamma_{j+1,-1} (\gamma_{i,1} Y_{j-1\ i+1} + \gamma_{i,-1} Y_{j-1\ i-1}) , \\
b_{0,4} &= c_{j+2} \gamma_{j+1,-1} Y_{j-1\ i} , \\
b_{1,2} &= \gamma_{j+1,-1} (\gamma_{j,-1} Y_{j-2\ i} + \gamma_{i,1} Y_{j-1\ i+1} + \gamma_{i,-1} Y_{j-1\ i-1}) \\
&\quad + \gamma_{i+1,1} \gamma_{i,1} Y_{ji+2} + \gamma_{i-1,-1} \gamma_{i,-1} Y_{ji-2} + (c_{i+1} + c_i - c_j) Y_{ji} , \\
b_{1,3} &= \gamma_{j+1,-1} (\gamma_{j,1} (\gamma_{i,1} Y_{ji+1} + \gamma_{i,-1} Y_{ji-1}) + \gamma_{j,-1} (\gamma_{i,1} Y_{j-2\ i+1} + \gamma_{i,-1} Y_{j-2\ i-1}) \quad (\text{A.6}) \\
&\quad + \gamma_{i+1,1} \gamma_{i,1} Y_{j-1\ i+2} + \gamma_{i-1,-1} \gamma_{i,-1} Y_{j-1\ i-2} + (c_{i+1} + c_i) Y_{j-1\ i}) , \\
b_{1,4} &= c_{j+2} \gamma_{j+1,-1} (\gamma_{j,-1} Y_{j-2\ i} + \gamma_{i,1} Y_{j-1\ i+1} + \gamma_{i,-1} Y_{j-1\ i-1}) , \\
b_{2,3} &= \gamma_{j+1,-1} \gamma_{j,-1} (\gamma_{j-1,1} \gamma_{j,1} Y_{ji} + (c_{i+1} + c_i - c_j) Y_{j-2\ i} \\
&\quad + \gamma_{i+1,1} \gamma_{i,1} Y_{j-2\ i+2} + \gamma_{i-1,-1} \gamma_{i,-1} Y_{j-2\ i-2}) , \\
b_{2,4} &= c_{j+2} \gamma_{j+1,-1} \gamma_{j,-1} (\gamma_{i,1} Y_{j-2\ i+1} + \gamma_{i,-1} Y_{j-2\ i-1} + \gamma_{j-1,1} Y_{j-1\ i}) , \\
b_{3,4} &= c_{j+1} c_{j+2} \gamma_{j+1,-1} \gamma_{j,-1} Y_{j-2\ i} .
\end{aligned}$$

From the first term in (A.5) and (A.6) we may then read off the contributions of the different operators in the $\frac{1}{4}$ -BPS multiple \mathcal{B}_{ji} where $\Delta = 2j + r + s - 1$ and $\ell = s - r - 1$.

For $i = j$ and $k = -1$, then, as shown in (4.37), (A.2) decomposes into contributions from two $\frac{1}{2}$ -BPS multiplets, $\gamma_{j,1}(\mathcal{G}_t(\mathcal{B}_{jj}) - \gamma_{j+1,1} \gamma_{j,1} \mathcal{G}_t(\mathcal{B}_{j+1\ j+1}))$. Detailed results can be easily obtained from (A.5) and (A.6) using for $n < m$ $Y_{nm} = -Y_{m-1\ n+1}$,

$$\mathcal{G}_{j+r}(\mathcal{B}_{jj}) = \sum_{s=r+1}^3 \hat{b}_{r,s} g_{1,j+s-1} - \sum_{s=0}^{r-1} \hat{b}_{s,r} g_{1,j+s-1} , \quad r = 0, 1, 2, \quad (\text{A.7})$$

where

$$\begin{aligned}
\hat{b}_{0,1} &= Y_{jj} , \quad \hat{b}_{0,2} = \gamma_{j,-1} Y_{jj-1} , \quad \hat{b}_{0,3} = \gamma_{j+1,-1} \gamma_{j,-1} Y_{j-1\ j-1} , \\
\hat{b}_{1,2} &= \gamma_{j,-1} \gamma_{j-1,-1} Y_{jj-2} , \quad \hat{b}_{1,3} = \gamma_{j+1,-1} \gamma_{j,-1} \gamma_{j-1,-1} Y_{j-1\ j-2} , \\
\hat{b}_{2,3} &= \gamma_{j+1,-1} \gamma_{j,-1}^2 \gamma_{j-1,-1} Y_{j-2\ j-2} .
\end{aligned} \quad (\text{A.8})$$

Hence the conformal partial wave expansion for a $\frac{1}{2}$ -BPS multiplet is determined by $\sum_{nm} a_{nm,j+r\ s-r}(\mathcal{B}_{jj}) Y_{nm} = \hat{b}_{r,s+1}$. For $j = 1$ $\hat{b}_{0,1} = Y_{11}$, $\hat{b}_{0,2} = \frac{1}{6} Y_{10}$, $\hat{b}_{0,3} = \frac{1}{30}$ and $\hat{b}_{1,2} = \hat{b}_{1,3} = \hat{b}_{2,3} = 0$, which is identical with (4.30).

Appendix B. Large N Free Field Results

We here calculate the leading large N behaviour for the free field contributions for the four point function for $[0, p, 0]$ chiral primary operators. The generators $\{T_a\}$, $a =$

$1, \dots, N^2$ of $U(N)$ are $N \times N$ matrices with the following properties

$$[T_a, T_b] = if_{abc}T_c, \quad \text{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}, \quad \text{tr}(T_a A) \text{tr}(T_a B) = \frac{1}{2}\text{tr}(AB), \quad T_a T_a = \frac{1}{2}N \mathbf{1}. \quad (\text{B.1})$$

For the purposes of the combinatorics of counting diagrams we define an adjoint scalar field with the basic two point function

$$\langle X_a X_b \rangle = 2\delta_{ab}. \quad (\text{B.2})$$

Defining the adjoint scalar $X = X_a T_a$ single trace chiral primary operators belonging to the $[0, p, 0]$ representation correspond to $\text{tr}(X^p)$.

We consider first the two-point function, using the identities (B.1),

$$\begin{aligned} \langle \text{tr}(X^p) \text{tr}(X^p) \rangle &= 2^p p! \text{tr}(T_{a_1} \dots T_{a_p}) \text{tr}(T_{a_1} \dots T_{a_p}) \\ &\simeq 2^p p \text{tr}(T_{a_1} \dots T_{a_p}) \text{tr}(T_{a_p} \dots T_{a_1}) \\ &= 2^{p-1} p \text{tr}(T_{a_1} \dots T_{a_{p-1}} T_{a_{p-1}} \dots T_{a_1}) = p N^p, \end{aligned} \quad (\text{B.3})$$

where in the second line, and also subsequently, \simeq denotes that sub-dominant terms for large N have been dropped. In the product of symmetrised traces of T_a 's, apart from cyclic permutations of the trace giving the factor p , only one arrangement of the T_a 's, as displayed in the second line, gives the leading large N behaviour leading to the factor N^p . The corresponding planar diagram has index loops $(12)^p$ each generating a factor N .

Corresponding to the three point function of three chiral primary operators similarly, taking into account symmetry factors with $p_1 = i + j$, $p_2 = j + k$, $p_3 = k + i$,

$$\begin{aligned} &\langle \text{tr}(X^{p_1}) \text{tr}(X^{p_2}) \text{tr}(X^{p_3}) \rangle \\ &= 2^{\frac{1}{2}(p_1+p_2+p_3)} p_1! p_2! p_3! \frac{1}{i!j!k!} \\ &\quad \times \text{tr}(T_{a_1} \dots T_{a_i} T_{b_1} \dots T_{b_j}) \text{tr}(T_{b_1} \dots T_{b_j} T_{c_1} \dots T_{c_k}) \text{tr}(T_{c_1} \dots T_{c_k} T_{a_1} \dots T_{a_i}) \\ &\simeq 2^{\frac{1}{2}(p_1+p_2+p_3)} p_1 p_2 p_3 \\ &\quad \times \text{tr}(T_{a_1} \dots T_{a_i} T_{b_1} \dots T_{b_j}) \text{tr}(T_{b_j} \dots T_{b_1} T_{c_1} \dots T_{c_k}) \text{tr}(T_{c_k} \dots T_{c_1} T_{a_i} \dots T_{a_1}) \\ &= 2^{\frac{1}{2}(p_1+p_2+p_3)-2} p_1 p_2 p_3 \\ &\quad \times \text{tr}(T_{a_1} \dots T_{a_i} T_{b_1} \dots T_{b_{j-1}} T_{b_{j-1}} \dots T_{b_1} T_{c_1} \dots T_{c_{k-1}} T_{c_{k-1}} \dots T_{c_1} T_{a_i} \dots T_{a_1}) \\ &= p_1 p_2 p_3 N^{\frac{1}{2}(p_1+p_2+p_3)-1}, \end{aligned} \quad (\text{B.4})$$

where again we keep only the ordering up to cyclic permutations, which corresponds to planar diagrams with index loops $(123)(132)(13)^{i-1}(12)^{j-1}(23)^{k-1}$ if $i, j, k > 0$. In the extremal case $p_1 + p_2 = p_3$ or $j = 0$, $p_1 = i$, $p_2 = k$ we have instead of (B.4)

$2^{p_3} p_3! \text{tr}(T_{(a_1 \dots T_{a_i})}) \text{tr}(T_{(c_1 \dots T_{c_k})}) \text{tr}(T_{(c_1 \dots T_{c_k} T_{a_1} \dots T_{a_i})})$ which for large N reduces to $2^{p_3-1} p_1! p_2 p_3 \text{tr}(T_{(a_1 \dots T_{a_i})}) \text{tr}(T_{c_1 \dots T_{c_{k-1}} T_{c_{k-1}} \dots T_{c_1} T_{a_i} \dots T_{a_1}) \simeq p_1 p_2 p_3 N^{p_3-1}$, so the final result is unchanged. The final result in (B.4) is of course identical to that of Lee *et al* [2].

For the four point function of four $\text{tr}(X^p)$ operators we define

$$\langle \text{tr}(X^p) \text{tr}(X^p) \text{tr}(X^p) \text{tr}(X^p) \rangle = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=p}} I_{ijk}, \quad (\text{B.5})$$

where I_{ijk} represents diagrams with i, j, k lines linking the external vertices, as shown in fig. 1.

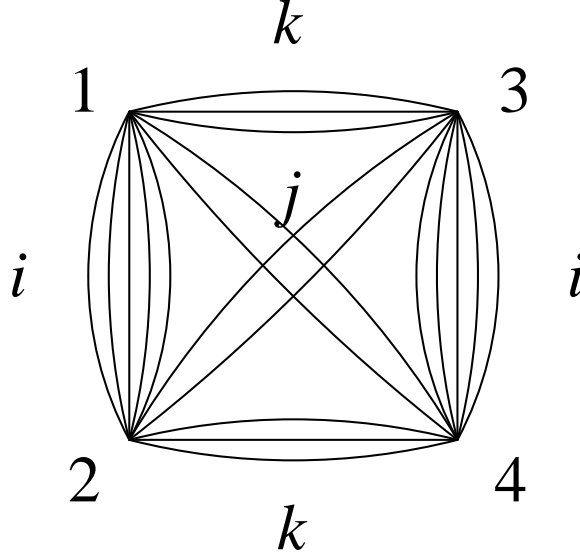


Fig. 1 Free field contributions to four point function for $i = 5, j = 2, k = 3$.

With appropriate symmetry factors we have

$$I_{ijk} = 2^{2p} \frac{p!^4}{(i!j!k!)^2} \text{tr}(T_{(a_1 \dots T_{a_i} T_{b_1} \dots T_{b_j} T_{c_1} \dots T_{c_k})}) \text{tr}(T_{(c_1 \dots T_{c_k} T_{d_1} \dots T_{d_j} T_{e_1} \dots T_{e_i})}) \\ \times \text{tr}(T_{(e_1 \dots T_{e_i} T_{b_1} \dots T_{b_j} T_{f_1} \dots T_{f_k})}) \text{tr}(T_{(f_1 \dots T_{f_k} T_{d_1} \dots T_{d_j} T_{a_1} \dots T_{a_i})}). \quad (\text{B.6})$$

If $i \leq j \leq k$ I_{ijk} and the associated permutations of i, j, k correspond to the symmetric polynomials $S_{ab}^{(p)}$ in (5.4) where $b = i, a = j$. For the disconnected diagrams then as in (B.3) we have

$$I_{p00} = I_{0p0} = I_{00p} = p^2 N^{2p}. \quad (\text{B.7})$$

If one of i, j, k are zero then we may reduce I_{ijk} to a single trace in the same fashion as

(B.4),

$$\begin{aligned}
I_{i0k} &\simeq 2^{2p-3} p^4 \operatorname{tr}(T_{a_1} \dots T_{a_i} T_{c_1} \dots T_{c_{k-1}} T_{c_{k-1}} \dots T_{c_1} T_{e_1} \dots T_{e_{i-1}} \\
&\quad \times T_{e_{i-1}} \dots T_{e_1} T_{f_1} \dots T_{f_{k-1}} T_{f_{k-1}} \dots T_{f_1} T_{a_i} \dots T_{a_1}) \\
&= p^4 N^{2p-2}.
\end{aligned} \tag{B.8}$$

In this case the index loops are $(1342)(1243)(13)^{k-1}(24)^{k-1}(12)^{i-1}(34)^{i-1}$. For $i, j, k > 0$ there are two possibilities

$$\begin{aligned}
I_{ijk} &\simeq 2^{2p-3} p^4 \left(\operatorname{tr}(T_{b_1} \dots T_{b_j} T_{a_1} \dots T_{a_i} T_{c_1} \dots T_{c_{k-1}} T_{c_{k-1}} \dots T_{c_1} T_{d_1} \dots T_{d_j} T_{e_1} \dots T_{e_{i-1}} \right. \\
&\quad \times T_{e_{i-1}} \dots T_{e_1} T_{f_1} \dots T_{f_{k-1}} T_{f_{k-1}} \dots T_{f_1} T_{d_j} \dots T_{d_1} T_{a_i} \dots T_{a_1} T_{b_j} \dots T_{b_1}) \\
&\quad + \operatorname{tr}(T_{d_1} \dots T_{d_j} T_{a_1} \dots T_{a_i} T_{b_1} \dots T_{b_j} T_{c_1} \dots T_{c_{k-1}} T_{c_{k-1}} \dots T_{c_1} T_{e_1} \dots T_{e_{i-1}} \\
&\quad \times T_{e_{i-1}} \dots T_{e_1} T_{b_j} \dots T_{b_1} T_{f_1} \dots T_{f_{k-1}} T_{f_{k-1}} \dots T_{f_1} T_{a_i} \dots T_{a_1} T_{d_j} \dots T_{d_1}) \Big) \\
&= 2p^4 N^{2p-2}.
\end{aligned} \tag{B.9}$$

These results correspond to two independent index loops for the associated planar diagrams $(124)(132)(234)(143)(13)^{k-1}(24)^{k-1}(12)^{i-1}(34)^{i-1}(14)^{j-1}(23)^{j-1}$ and also its reverse $(123)(134)(214)(243)(13)^{k-1}(24)^{k-1}(12)^{i-1}(34)^{i-1}(14)^{j-1}(23)^{j-1}$. The results (B.8) and (B.9) justify (5.9), normalising the disconnected contribution to 1 by dividing by $p^2 N^{2p}$.

Appendix C. Proof of Hypergeometric Identities

To demonstrate the first result in (9.3) we use a standard integral representation for hypergeometric functions,

$$x^{N+2} f_{n, N+2, n+1, N+1}(1-x) = (n-1)! (N+1)! \int_0^1 t^n (1-t)^N \left(\frac{x}{1-tx} \right)^{N+2} dt. \tag{C.1}$$

Integrating by parts $N+1$ times gives

$$\begin{aligned}
x^{N+2} f_{n, N+2, n+1, N+1}(1-x) &= -(n-1)! N! \left(x' + (-1)^n \sum_{s=0}^{N-n} \frac{(N-s)! s!}{(N-n-s)! (s+n)!} x^{s+1} \right) \\
&\quad - (-1)^N (n-1)! \int_0^1 \frac{d^{N+1}}{dt^{N+1}} (t^n (1-t)^N) \frac{x}{1-tx} dt.
\end{aligned} \tag{C.2}$$

In the second line we may use the identity (3.12) to then obtain (9.3) with

$$r_\ell^{(n, N)} = (-1)^{\ell+N} \frac{\ell!^2}{(2\ell)!} (n-1)! \int_0^1 \frac{d^{N+1}}{dt^{N+1}} (t^n (1-t)^N) P_\ell(2t-1) dt, \tag{C.3}$$

where it is easy to see that $r_\ell^{(n,N)} = 0$ for $\ell \geq n$. The second relation in (9.3) follows directly using the symmetry under $x \rightarrow x'$. There is no simple evaluation for $r_\ell^{(n,N)}$ but by integration by parts, using $P_\ell(2t-1) = (-1)^\ell \frac{1}{\ell!} \frac{d^\ell}{dt^\ell} (t(1-t))^\ell$, we may find for $\ell = 0, \dots, n-1, n = 1, \dots, N$,

$$\begin{aligned} r_\ell^{(n,N)} &= (-1)^\ell \frac{1}{\ell+1} \frac{\ell!}{(2\ell)!} \frac{(n-1)!^2 n! (N+\ell+1)!}{(\ell+n)! (n-\ell-1)!} {}_3F_2(\ell+1-n, -N, \ell+1; 1-n, \ell+2; 1) \\ &= \frac{\ell!^2}{(2\ell)!} (n-1)! N! \left((-1)^\ell - (-1)^n \binom{N}{n} {}_3F_2(-\ell, n-N, \ell+1; -N, n+1; 1) \right). \end{aligned} \quad (C.4)$$

Note that $r_\ell^{(1,N)} = (N+1)! \delta_{\ell 0}$. The second formula is sufficient to verify, using (3.11), that the leading terms in an expansion in powers of x on the right hand side of (9.3) are cancelled.

To discuss the expansion of $\mathcal{F}_{00}(v)$ as in (9.1) we use the same integral representation as in (C.1) to first write

$$f_{1322}(v) = 2 \int_0^1 \frac{t(1-t)}{(1-tx-t(1-x)\bar{x})^3} dt = \frac{2}{1-x} \int_0^1 \frac{z(1-z)}{1-(1-z)x} \frac{1}{(1-z\bar{x})^3} dz. \quad (C.5)$$

For application in (9.1) we use

$$2u^2 \frac{x-\bar{x}}{(1-z\bar{x})^3} = -x^2 \left((1-xz) \frac{d^2}{dz^2} - 2x \frac{d}{dz} \right) \frac{\bar{x}}{1-z\bar{x}}. \quad (C.6)$$

The expansion (3.12) and (9.1) then gives

$$\begin{aligned} \mathcal{F}_{00,j}(x) &= \frac{j!^2}{(2j)!} x x' \int_0^1 \frac{z(1-z)}{1-(1-z)x} \left((1-xz) \frac{d^2}{dz^2} - 2x \frac{d}{dz} \right) P_j(2z-1) dz \\ &= -\frac{j!^2}{(2j)!} \int_0^1 \frac{x^2}{(1-(1-z)x)^2} (2z-1) \frac{d}{dz} P_j(2z-1) dz \\ &= \frac{j!^2}{(2j)!} j(j+1) (x + (-1)^j x') \\ &\quad - \frac{j!^2}{(2j)!} \int_0^1 \frac{x}{1-(1-z)x} \frac{d}{dz} (2z-1) \frac{d}{dz} P_j(2z-1) dz. \end{aligned} \quad (C.7)$$

Inserting the expansion (3.12) once more gives (9.5) with

$$c_{j,\ell} = \int_0^1 P_\ell(2z-1) \frac{d}{dz} (2z-1) \frac{d}{dz} P_j(2z-1) dz. \quad (C.8)$$

It is easy to see that $c_{j,\ell} = 0$ for $\ell \geq j$. The integral may be evaluated to give (9.6).

More generally we may extend (C.5) to

$$f_{nN+2n+1N+1}(v) = (n-1)!(N+1)! \frac{1}{1-x} \int_0^1 \frac{z^n(1-z)^N}{(1-(1-z)x)^n} \frac{1}{(1-z\bar{x})^{N+2}} dz, \quad (\text{C.9})$$

and then obtain

$$\begin{aligned} (x - \bar{x}) u^{N+1} f_{nN+2n+1N+1}(v) &= (n-1)! \int_0^1 f^{(n,N)}(x, z) \frac{d^N}{dz^N} \frac{\bar{x}}{1 - z\bar{x}} dz \\ &= -(-1)^j \sum_j f_j^{(n,N)}(x) g_{1,j+1}(\bar{x}), \end{aligned} \quad (\text{C.10})$$

where

$$f^{(n,N)}(x, z) = \frac{x^{N+1} z^{n-1} (1-z)^{N-1}}{(1-(1-z)x)^{n+1}} (n(1-2z) + (n-N)z(1-(1-z)x)). \quad (\text{C.11})$$

$f_j^{(n,N)}(x)$ may be determined by using (3.12),

$$f_j^{(n,N)}(x) = (n-1)! \frac{j!^2}{(2j)!} \int_0^1 f^{(n,N)}(x, z) \frac{d^N}{dz^N} P_j(2z-1) dz, \quad (\text{C.12})$$

which is non zero for $j \geq N$. To evaluate this we express $f^{(n,N)}(x)$ in partial fractions in the form

$$\begin{aligned} f^{(n,N)}(x, z) &= (-1)^n \left(\sum_{r=0}^n \alpha_r(z) \frac{1}{r!} \frac{d^r}{dz^r} \frac{x}{1-(1-z)x} - \sum_{s=0}^{N-n} \beta_s(z) x^{s+1} \right), \\ \alpha_r(z) &= \binom{N-r-1}{n-r} z^{n-1} (1-z)^{r-1} (n(1-z) - rz), \\ \beta_s(z) &= \binom{N-s-1}{n-1} z^{n-1} (1-z)^{s-1} (s(2z-1) + (N-n)(1-z)), \end{aligned} \quad (\text{C.13})$$

and then using (3.12) and integrating by parts $f_j^{(n,N)}(x)$ is determined as a polynomial in x and x' up to a linear combination of $g_{1,\ell+1}(x)$ for $\ell = 0, \dots, j-N-n+1$ if $j \geq N+n-1$.

As particular cases we have

$$\begin{aligned} f^{(2,2)}(x, z) &= z(1-z)(1-2z) \frac{d^2}{dz^2} \frac{x}{1-(1-z)x}, \\ f^{(1,2)}(x, z) &= - \left((1-2z) \frac{d}{dz} + 1 \right) \frac{x}{1-(1-z)x} + zx^2 + x, \end{aligned} \quad (\text{C.14})$$

and hence, neglecting terms involving $g_{1,\ell+1}(x)$,

$$\begin{aligned}
f_j^{(2,2)}(x) &\sim -\frac{j!^2}{(2j)!} \frac{1}{2}(j-1)j(j+1)(j+2)(x + (-1)^j x') , \\
f_j^{(1,2)}(x) &\sim \frac{j^2}{(2j)!} \left(\frac{1}{2}(j-1)j(j+1)(j+2)(x - (-1)^j x') \right. \\
&\quad \left. + j(j+1)(x^2 + x + (-1)^j x) - x^2(1 - (-1)^j) \right) .
\end{aligned} \tag{C.15}$$

Since $\mathcal{F}_{10}(v) = f_{2433}(v) + f_{1423}(v)$ from (9.1) we have $\mathcal{F}_{10,j}(x) = f_j^{(2,2)}(x) + f_j^{(1,2)}(x)$ and the result obtained from (C.15) coincides with a direct calculation of $\mathcal{K}_{10,j}(x)$ as in section 5.

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